Corrigé du devoir maison n°5

Diagonale d'un pentagone régulier

1. (a) On a
$$S = a + a^4 = e^{\frac{2i\pi}{5}} + (e^{\frac{2i\pi}{5}})^4 = e^{\frac{2i\pi}{5}} + e^{\frac{8i\pi}{5}} = e^{\frac{2i\pi}{5}} + e^{\frac{-2i\pi}{5}}$$
donc $\boxed{S = 2\cos\left(\frac{2\pi}{5}\right)}$ et
$$T = a^2 + a^3 = (e^{\frac{2i\pi}{5}})^2 + (e^{\frac{2i\pi}{5}})^3 = e^{\frac{4i\pi}{5}} + e^{\frac{6i\pi}{5}} = e^{\frac{4i\pi}{5}} + e^{\frac{-4i\pi}{5}}$$
d'où $\boxed{T = 2\cos\left(\frac{4\pi}{5}\right).}$

(b) On a $S + T = a + a^2 + a^3 + a^4$. On reconnaît la somme de termes consécutifs d'une progression géométrique avec $a \neq 1$ donc
$$S + T = \sum_{k=1}^{4} a^k = a\frac{1-a^4}{1-a} = e^{\frac{2i\pi}{5}}\frac{1 - e^{\frac{-2i\pi}{5}}}{1 - e^{\frac{2i\pi}{5}}} = \frac{e^{\frac{2i\pi}{5}} - 1}{1 - e^{\frac{2i\pi}{5}}}$$
d'où $\boxed{S + T = -1}$.
Par ailleurs, on a
$$ST = (a + a^4)(a^2 + a^3) = a^3 + a^4 + a^6 + a^7.$$
Or, on remarque que $a^6 = e^{\frac{12i\pi}{5}} = e^{\frac{2i\pi}{5}} = a$ et $a^7 = a^6 \times a = a \times a = a^2$ donc $ST = a + a^2 + a^3 + a^4 = S + T$ d'où $\boxed{ST = -1}$.

(c) S et T sont racines du trinôme du second degré $x^2 - (S+T)x + ST = 0$, i.e.
$$\boxed{x^2 + x - 1 = 0.}$$

(d) Calculons les racines du trinôme $x^2 + x - 1 = 0$. On a $\Delta = 1 - 4 \times 1 \times (-1) = 5 > 0$ donc les racines sont
$$x_1 = \frac{-1-\sqrt{5}}{2} < 0 \quad \text{et} \quad x_2 = \frac{-1+\sqrt{5}}{2} > 0.$$
Or, on sait que les racines sont $S = 2\cos\left(\frac{2\pi}{5}\right)$ et $T = 2\cos\left(\frac{4\pi}{5}\right)$.
Puisque $\frac{2\pi}{5} \in]0, \frac{\pi}{2}[$, $\cos\left(\frac{2\pi}{5}\right) > 0$ et puisque $\frac{4\pi}{5} \in]\frac{\pi}{2}, \pi[$, $\cos\left(\frac{4\pi}{5}\right) < 0$.
On en déduit que $S = x_2$ et $T = x_1$ donc $2\cos\left(\frac{2\pi}{5}\right) = \frac{-1+\sqrt{5}}{2}$ d'où $\boxed{\cos\left(\frac{2\pi}{5}\right) = \frac{\sqrt{5}-1}{4}.}$
Enfin, d'après la relation fondamentale de la trigonométrie, $\cos^2\left(\frac{2\pi}{5}\right) + \sin^2\left(\frac{2\pi}{5}\right) = 1$ donc
$$\sin^2\left(\frac{2\pi}{5}\right) = 1 - \cos^2\left(\frac{2\pi}{5}\right) = 1 - \frac{5 - 2\sqrt{5} + 1}{16} = \frac{10 + 2\sqrt{5}}{16} = \frac{5+\sqrt{5}}{8}.$$

Puisque $\frac{2\pi}{5} \in]0,\pi[, \sin\left(\frac{2\pi}{5}\right) > 0$ donc

$$\sin\left(\frac{2\pi}{5}\right) = \sqrt{\frac{5+\sqrt{5}}{8}} = \frac{\sqrt{5+\sqrt{5}}}{2\sqrt{2}}$$

d'où en multipliant par $\sqrt{2}$ au numérateur et au dénominateur, $\boxed{\sin\left(\frac{2\pi}{5}\right) = \frac{\sqrt{10+2\sqrt{5}}}{4}.}$

2. (a) Soit $k \in [\![0,4]\!]$. On a

$$\boxed{z_k^5 = (e^{\frac{2ik\pi}{5}})^5 = e^{2ik\pi} = 1.}$$

Par ailleurs, $\displaystyle\sum_{k=0}^{4} z_k = \sum_{k=0}^{4}\left(e^{\frac{2ik\pi}{5}}\right) = \sum_{k=0}^{4}\left(e^{\frac{2i\pi}{5}}\right)^k = \sum_{k=0}^{4} a^k = 1 + \sum_{k=1}^{4} a^k.$

Or, d'après la question 1.b), $\displaystyle\sum_{k=1}^{4} a^k = S + T = -1$ donc $\displaystyle 1 + \sum_{k=1}^{4} a^k = 0.$

On en déduit que $\boxed{\displaystyle\sum_{k=0}^{4} z_k = 0.}$

(b) Pour tout $k \in [\![0,3]\!]$, on a

$$M_kM_{k+1} = |z_{k+1}-z_k| = |e^{\frac{2i\pi(k+1)}{5}} - e^{\frac{2ik\pi}{5}}| = |e^{\frac{2ik\pi}{5}}(e^{\frac{2i\pi}{5}}-1)| = |e^{\frac{2ik\pi}{5}}||e^{\frac{2i\pi}{5}}-1| = |e^{\frac{2i\pi}{5}}-1|$$

car $|e^{\frac{2ik\pi}{5}}| = 1$ donc $M_0M_1 = M_1M_2 = M_2M_3 = M_3M_4 = |e^{\frac{2i\pi}{5}} - 1|$ et

$$|e^{\frac{2i\pi}{5}} - 1| = |e^{\frac{i\pi}{5}}(e^{\frac{i\pi}{5}} - e^{-\frac{i\pi}{5}})| = |e^{\frac{i\pi}{5}}||2i\sin\left(\frac{\pi}{5}\right)| = 2\sin\left(\frac{\pi}{5}\right).$$

De même,

$$M_4M_0 = |z_0-z_4| = |1-(e^{\frac{2i\pi}{5}})^4| = |1-e^{\frac{8i\pi}{5}}| = |1-e^{-\frac{2i\pi}{5}}| = |e^{-\frac{2i\pi}{5}}(e^{\frac{2i\pi}{5}}-1)| = |e^{-\frac{2i\pi}{5}}||e^{\frac{2i\pi}{5}}-1|$$

d'où $M_4M_0 = |e^{\frac{2i\pi}{5}} - 1|$ donc $\boxed{M_4M_0 = M_0M_1 = M_1M_2 = M_2M_3 = M_3M_4 = L}$ avec $L = 2\sin\left(\frac{\pi}{5}\right)$.

Pour tout $k \in [\![0,3]\!]$, on a

$$(\widehat{\overrightarrow{OM_k}, \overrightarrow{OM_{k+1}}}) = (\widehat{\vec{i}, \overrightarrow{OM_{k+1}}}) - (\widehat{\vec{i}, \overrightarrow{OM_k}}) \equiv \arg(z_{k+1}) - \arg(z_k) \equiv \frac{2(k+1)\pi}{5} - \frac{2k\pi}{5} \equiv \frac{2\pi}{5}[2\pi]$$

et

$$(\widehat{\overrightarrow{OM_4}, \overrightarrow{OM_0}}) = (\widehat{\vec{i}, \overrightarrow{OM_0}}) - (\widehat{\vec{i}, \overrightarrow{OM_4}}) \equiv \arg(z_0) - \arg(z_4) \equiv 0 - \frac{8\pi}{5} \equiv \frac{2\pi}{5}[2\pi].$$

Ainsi,

$$\boxed{(\widehat{\overrightarrow{OM_0}, \overrightarrow{OM_1}}) = (\widehat{\overrightarrow{OM_1}, \overrightarrow{OM_2}}) = (\widehat{\overrightarrow{OM_2}, \overrightarrow{OM_3}}) = (\widehat{\overrightarrow{OM_3}, \overrightarrow{OM_4}}) = (\widehat{\overrightarrow{OM_4}, \overrightarrow{OM_0}}) = \frac{2\pi}{5}.}$$

Enfin, pour tout couple de vecteurs $(\vec{u}, \vec{v})$, on a

$$(\widehat{\vec{u}, \vec{v}}) = (\widehat{\vec{i}, \vec{v}}) - (\widehat{\vec{i}, \vec{u}}) \equiv \arg(z_{\vec{v}}) - \arg(z_{\vec{u}})[2\pi]$$

où $z_{\vec{u}}$ et $z_{\vec{v}}$ sont les affixes respectives de $\vec{u}$ et $\vec{v}$.
Ainsi, pour tout $k \in [\![1,3]\!]$,

$$(\widehat{\overrightarrow{M_kM_{k+1}}, \overrightarrow{M_kM_{k-1}}}) \equiv \arg(z_{k-1}-z_k) - \arg(z_{k+1}-z_k) \equiv \arg\left(\frac{z_{k-1}-z_k}{z_{k+1}-z_k}\right) [2\pi].$$

Or, on a

$$\frac{z_{k-1}-z_k}{z_{k+1}-z_k} = \frac{e^{\frac{2i(k-1)\pi}{5}} - e^{\frac{2ik\pi}{5}}}{e^{\frac{2i(k+1)\pi}{5}} - e^{\frac{2ik\pi}{5}}} = \frac{e^{\frac{2i(k-1)\pi}{5}}}{e^{\frac{2ik\pi}{5}}} \frac{1-e^{\frac{2i\pi}{5}}}{e^{\frac{2i\pi}{5}}-1} = -\frac{1}{e^{\frac{2i\pi}{5}}} = \frac{e^{i\pi}}{e^{\frac{2i\pi}{5}}} = e^{\frac{3i\pi}{5}}.$$

On en déduit que pour tout $k \in [\![1,3]\!]$, $(\widehat{\overrightarrow{M_kM_{k+1}}, \overrightarrow{M_kM_{k-1}}}) = \dfrac{3\pi}{5}$.
De même,

$$(\widehat{\overrightarrow{M_0M_1}, \overrightarrow{M_0M_4}}) \equiv \arg\left(\frac{z_4-z_0}{z_1-z_0}\right) \equiv \arg\left(\frac{e^{\frac{8i\pi}{5}}-1}{e^{\frac{2i\pi}{5}}-1}\right) \equiv \arg\left(\frac{e^{-\frac{2i\pi}{5}}-1}{e^{\frac{2i\pi}{5}}-1}\right).$$

Or, $\dfrac{e^{-\frac{2i\pi}{5}}-1}{e^{\frac{2i\pi}{5}}-1} = e^{-\frac{2i\pi}{5}} \dfrac{1-e^{\frac{2i\pi}{5}}}{e^{\frac{2i\pi}{5}}-1} = -e^{-\frac{2i\pi}{5}} = e^{i\pi}e^{-\frac{2i\pi}{5}} = e^{\frac{3i\pi}{5}}$ donc

$$(\widehat{\overrightarrow{M_0M_1}, \overrightarrow{M_0M_4}}) = \frac{3\pi}{5}.$$

Enfin,

$$(\widehat{\overrightarrow{M_4M_0}, \overrightarrow{M_4M_3}}) \equiv \arg\left(\frac{z_3-z_4}{z_0-z_4}\right) \equiv \arg\left(\frac{e^{\frac{6i\pi}{5}}-e^{\frac{8i\pi}{5}}}{1-e^{\frac{8i\pi}{5}}}\right) \equiv \arg\left(e^{-\frac{2i\pi}{5}} \frac{e^{\frac{8i\pi}{5}}-1}{1-e^{\frac{8i\pi}{5}}}\right) \equiv \arg(e^{-\frac{2i\pi}{5}}e^{i\pi}),$$

donc on trouve encore $(\widehat{\overrightarrow{M_4M_0}, \overrightarrow{M_4M_3}}) = \dfrac{3\pi}{5}$.
On a donc bien montré que

$$\boxed{(\widehat{\overrightarrow{M_0M_1}, \overrightarrow{M_0M_4}}) = (\widehat{\overrightarrow{M_1M_2}, \overrightarrow{M_1M_0}}) = (\widehat{\overrightarrow{M_2M_3}, \overrightarrow{M_2M_1}}) = (\widehat{\overrightarrow{M_3M_4}, \overrightarrow{M_3M_2}}) = (\widehat{\overrightarrow{M_4M_0}, \overrightarrow{M_4M_3}}) = \frac{3\pi}{5}.}$$

(c) Pour tout $k \in [\![0,2]\!]$, on a

$$M_kM_{k+2} = |z_{k+2}-z_k| = |e^{\frac{2i(k+2)\pi}{5}} - e^{\frac{2ik\pi}{5}}| = |e^{\frac{2ik\pi}{5}}(e^{\frac{4i\pi}{5}}-1)| = |e^{\frac{2ik\pi}{5}}||e^{\frac{4i\pi}{5}}-1| = |e^{\frac{4i\pi}{5}}-1|.$$

De même, $M_3M_0 = |z_3 - z_0| = |e^{\frac{6i\pi}{5}} - 1| = |e^{-\frac{4i\pi}{5}} - 1| = |\overline{e^{\frac{4i\pi}{5}} - 1}| = |e^{\frac{4i\pi}{5}} - 1|$.
Enfin, $M_4M_1 = |z_1 - z_4| = |e^{\frac{2i\pi}{5}} - e^{\frac{8i\pi}{5}}| = |e^{\frac{2i\pi}{5}}||1 - e^{\frac{6i\pi}{5}}| = |e^{\frac{6i\pi}{5}} - 1| = |e^{\frac{4i\pi}{5}} - 1|$. Or, on a

$$|e^{\frac{4i\pi}{5}} - 1| = |e^{\frac{2i\pi}{5}}(e^{\frac{4i\pi}{5}} - e^{\frac{-2i\pi}{5}})| = |e^{\frac{2i\pi}{5}}||2i\sin\left(\frac{2\pi}{5}\right)| = 2\sin\left(\frac{2\pi}{5}\right)$$

En posant $D = 2\sin\left(\dfrac{2\pi}{5}\right)$, on a donc bien

$$\boxed{M_0M_2 = M_1M_3 = M_2M_4 = M_3M_0 = M_4M_1 = D.}$$

Enfin, calculons $\dfrac{D}{L}$. On a

$$\frac{D}{L} = \frac{2\sin\left(\frac{2\pi}{5}\right)}{2\sin\left(\frac{\pi}{5}\right)} = \frac{\sin\left(\frac{2\pi}{5}\right)}{\sin\left(\frac{\pi}{5}\right)} = \frac{2\sin\left(\frac{\pi}{5}\right)\cos\left(\frac{\pi}{5}\right)}{\sin\left(\frac{\pi}{5}\right)} = 2\cos\left(\frac{\pi}{5}\right) = 2\cos\left(\pi - \frac{4\pi}{5}\right) = -2\cos\left(\frac{4\pi}{5}\right).$$

Or, d'après la question 1.(d), $2\cos\left(\dfrac{4\pi}{5}\right) = T = \dfrac{-1-\sqrt{5}}{2}$.

Ainsi, $\dfrac{D}{L} = \dfrac{1+\sqrt{5}}{2}$. On reconnaît la valeur du nombre d'or φ.

On en conclut que $\boxed{\dfrac{D}{L} = \varphi.}$

(d) Si la longueur du pentagone L vaut 1, on en déduit que la longueur de la diagonale vaut $\boxed{D = \varphi.}$