

Corrigé de la liste d'exercices n°3

Méthodes de calcul

Exercice 1

$$1. \sum_{i=1}^n 1 = n.$$

$$2. \sum_{i=1}^n \sum_{j=1}^n 1 = \sum_{i=1}^n n = n^2.$$

$$3. \sum_{i=1}^n \sum_{j=1}^i 1 = \sum_{i=1}^n i = \frac{n(n+1)}{2}.$$

$$4. \sum_{i=1}^n \sum_{j=1}^n (i+j) = \sum_{i=1}^n \left(\sum_{j=1}^n i + \sum_{j=1}^n j \right) = \sum_{i=1}^n \left(n \times i + \frac{n(n+1)}{2} \right) = \frac{n^2(n+1)}{2} + \frac{n^2(n+1)}{2} = n^2(n+1).$$

$$5. \sum_{i=1}^n \sum_{j=i}^n j = \sum_{j=1}^n \sum_{i=1}^j j = \sum_{j=1}^n j^2 = \frac{n(n+1)(2n+1)}{6}.$$

$$6. \sum_{k=0}^n \sum_{j=1}^m kj^2 = \left(\sum_{k=0}^n k \right) \left(\sum_{j=1}^m j^2 \right) = \frac{n(n+1)m(m+1)(2m+1)}{12}.$$

7.

$$\begin{aligned} \sum_{k=0}^{n^2} \sum_{j=k}^{k+2} kj^2 &= \sum_{k=0}^{n^2} k(k^2 + (k+1)^2 + (k+2)^2) \\ &= \sum_{k=0}^{n^2} 3k^3 + 6k^2 + 5k \\ &= 3 \frac{n^4(n^2+1)^2}{4} + n^2(n^2+1)(2n^2+1) + \frac{5n^2(n^2+1)}{2} \\ &= \frac{n^2(n^2+1)}{4} (3n^2(n^2+1) + 8n^2 + 4 + 10) \\ &= \frac{n^2(n^2+1)(3n^4 + 11n^2 + 14)}{4} \end{aligned}$$

Exercice 2

$$1. \sum_{k=0}^n x^{2k+1} = x \sum_{k=0}^n (x^2)^k = \begin{cases} n+1 & \text{si } x = 1 \\ -n-1 & \text{si } x = -1 \\ x \times \frac{1-x^{2n+2}}{1-x^2} & \text{si } x \notin \{-1, 1\}. \end{cases}$$

$$2. \sum_{k=0}^n a^k 2^{3k} x^{-k} = \sum_{k=0}^n (8ax^{-1})^k = \begin{cases} n+1 & \text{si } x = 8a \\ \frac{1-(8ax^{-1})^{n+1}}{1-8ax^{-1}} & \text{si } x \neq 8a. \end{cases}$$

$$3. \text{ Si } x = 1, \text{ on a } \sum_{k=0}^n \sum_{j=0}^k x^j = \sum_{k=0}^n (k+1) = \sum_{k=1}^{n+1} k = \frac{(n+1)(n+2)}{2}.$$

Supposons dorénavant que $x \neq 1$.

$$\begin{aligned} \sum_{k=0}^n \sum_{j=0}^k x^j &= \sum_{k=0}^n \frac{1 - x^{k+1}}{1 - x} \\ &= \frac{1}{1 - x} \left(n + 1 - \sum_{k=0}^n x^{k+1} \right) \\ &= \frac{1}{1 - x} \left(n + 1 - x \sum_{k=0}^n x^k \right) \\ &= \frac{1}{1 - x} \left(n + 1 - x \times \frac{1 - x^{n+1}}{1 - x} \right). \end{aligned}$$

$$4. \sum_{k=1}^n (-1)^k 2^{2k+1} = 2 \sum_{k=1}^n (-4)^k = 2 \times (-4) \times \frac{1 - (-4)^n}{1 - (-4)} = \frac{8}{5}((-4)^n - 1).$$

$$5. \sum_{k=1}^n 2^{2k+1} 3^{2k-1} = \frac{2}{3} \sum_{k=1}^n 36^k = \frac{2}{3} \times 36 \times \frac{1 - 36^n}{1 - 36} = \frac{24}{35}(36^n - 1).$$

$$6. \sum_{k=2}^{n+1} \frac{1}{(1-k)^3} + \frac{1}{k^3} = \sum_{k=2}^{n+1} \frac{1}{k^3} - \frac{1}{(k-1)^3} = \frac{1}{(n+1)^3} - 1.$$

$$7. \sum_{k=1}^n \frac{k}{(k+1)!} = \sum_{k=1}^n \frac{k+1-1}{(k+1)!} = \sum_{k=1}^n \frac{1}{k!} - \frac{1}{(k+1)!} = 1 - \frac{1}{(n+1)!}.$$

Exercice 3

$$1. S_n = \sum_{k=0}^n (-2a)^k = \begin{cases} n+1 & \text{si } a = -\frac{1}{2}. \\ \frac{1 - (-2a)^{n+1}}{1 + 2a} & \text{si } a \neq -\frac{1}{2}. \end{cases}$$

$$2. T_n = \sum_{k=0}^{n-1} (e^{\frac{a}{n}})^k = \begin{cases} n & \text{si } a = 0. \\ \frac{1 - e^a}{1 - e^{\frac{a}{n}}} & \text{si } a \neq 0. \end{cases}$$

Exercice 4

On a pour tout $n \in \mathbb{N}$;

$$\sum_{k=0}^n (2k+1) = 2 \sum_{k=0}^n k + \sum_{k=0}^n 1 = n(n+1) + n + 1 = (n+1)^2.$$

Exercice 5

On pose pour tout $n \in \mathbb{N}$, $u_n = \frac{1}{2^n}$.

1.

$$\sum_{k=0}^{2n+1} u_k = \sum_{k=0}^{2n+1} \frac{1}{2^k} = \sum_{k=0}^{2n+1} \left(\frac{1}{2}\right)^k = \frac{1 - \left(\frac{1}{2}\right)^{2n+2}}{1 - \frac{1}{2}} = 2 \left(1 - \frac{1}{2^{2n+2}}\right).$$

2.

$$\sum_{k=1}^{2n} u_{2k} = \sum_{k=1}^{2n} \frac{1}{2^{2k}} = \sum_{k=1}^{2n} \frac{1}{4^k} = \frac{1}{4} \frac{1 - \frac{1}{4^{2n}}}{1 - \frac{1}{4}} = \frac{1}{3} \left(1 - \frac{1}{4^{2n}}\right).$$

3. En posant le changement d'indice $i = 2n - k$, on a

$$\sum_{k=0}^n u_{2n-k} = \sum_{i=n}^{2n} u_i = \sum_{i=n}^{2n} \frac{1}{2^i} = \frac{1}{2^n} \frac{1 - \frac{1}{2^{n+1}}}{1 - \frac{1}{2}} = \frac{1}{2^{n-1}} \left(1 - \frac{1}{2^{n+1}}\right).$$

4. En posant le changement d'indice $i = k + n$, on obtient

$$\sum_{k=0}^n u_{k+n} = \sum_{i=n}^{2n} u_i = \frac{1}{2^{n-1}} \left(1 - \frac{1}{2^{n+1}}\right).$$

5.

$$\sum_{k=0}^n u_k + n = \sum_{k=0}^n \frac{1}{2^k} + n = \frac{1 - \frac{1}{2^{n+1}}}{1 - \frac{1}{2}} + n = 2 \left(1 - \frac{1}{2^{n+1}}\right) + n.$$

Exercice 6

On a

$$(a - b) \sum_{k=0}^{n-1} a^k b^{n-1-k} = \sum_{k=0}^{n-1} a^{k+1} b^{n-1-k} - \sum_{k=0}^{n-1} a^k b^{n-k}.$$

Dans la première somme, on pose le changement d'indice $i = k + 1$ et on obtient

$$(a - b) \sum_{k=0}^{n-1} a^k b^{n-1-k} = \sum_{i=1}^n a^i b^{n-i} - \sum_{k=0}^{n-1} a^k b^{n-k} = \sum_{k=1}^{n-1} a^k b^{n-k} + a^n - \sum_{k=1}^{n-1} a^k b^{n-k} - b^n = a^n - b^n.$$

Exercice 7

1. (a) Par linéarité de la somme, on a

$$\begin{aligned} \sum_{k=1}^n (k-1)^2 &= \sum_{k=1}^n (k^2 - 2k + 1) \\ &= \sum_{k=1}^n k^2 - 2 \sum_{k=1}^n k + \sum_{k=1}^n 1 \\ &= \frac{n(n+1)(2n+1)}{6} - n(n+1) + n \\ &= \frac{n}{6} (2n^2 + 3n + 1 - 6n) \\ &= \frac{n}{6} (2n^2 - 3n + 1) \\ &= \frac{(n-1)n(2n-1)}{6}. \end{aligned}$$

(b) On pose $i = k - 1$ et on obtient

$$\sum_{k=1}^n (k-1)^2 = \sum_{i=0}^{n-1} i^2 = \frac{(n-1)n(2(n-1)+1)}{6} = \frac{(n-1)n(2n-1)}{6}.$$

2. (a) Pour tout $k \in \llbracket 1, n \rrbracket$, on a

$$(k+1)^4 - k^4 = ((k+1)^2 - k^2)((k+1)^2 + k^2) = (2k+1)(2k^2 + 2k + 1) = 4k^3 + 6k^2 + 4k + 1.$$

(b) D'après la question précédente, on a

$$\begin{aligned} \sum_{k=0}^n (k+1)^4 - k^4 &= \sum_{k=0}^n (4k^3 + 6k^2 + 4k + 1) \\ \Leftrightarrow (n+1)^4 &= 4 \sum_{k=0}^n k^3 + 6 \sum_{k=0}^n k^2 + 4 \sum_{k=0}^n k + \sum_{k=0}^n 1 \\ \Leftrightarrow (n+1)^4 &= 4 \sum_{k=0}^n k^3 + n(n+1)(2n+1) + 2n(n+1) + (n+1) \\ \Leftrightarrow 4 \sum_{k=0}^n k^3 &= (n+1)[(n+1)^3 - 2n^2 - n - 2n - 1] \\ \Leftrightarrow 4 \sum_{k=0}^n k^3 &= (n+1)(n^3 + n^2) \\ \Leftrightarrow 4 \sum_{k=0}^n k^3 &= n^2(n+1)^2 \end{aligned}$$

$$\text{donc } \sum_{k=0}^n k^3 = \frac{n^2(n+1)^2}{4} = \left(\frac{n(n+1)}{2}\right)^2 = \left(\sum_{k=0}^n k\right)^2.$$

Exercice 8

1. En utilisant la séparation des variables, on a

$$\begin{aligned} \sum_{\substack{0 \leq i \leq n \\ 0 \leq j \leq p}} i(j^2 + 1) &= \left(\sum_{i=0}^n i\right) \left(\sum_{j=0}^p (j^2 + 1)\right) \\ &= \frac{n(n+1)}{2} \left(\sum_{j=0}^p j^2 + \sum_{j=0}^p 1\right) \\ &= \frac{n(n+1)}{2} \left(\frac{p(p+1)(2p+1)}{6} + p + 1\right) \\ &= \frac{n(n+1)}{12} (p+1)(p(2p+1) + 6) \\ &= \frac{n(n+1)}{12} (p+1)(2p^2 + p + 6). \end{aligned}$$

2. En utilisant la séparation des variables, on a

$$\begin{aligned}\sum_{0 \leq i, j \leq n} 2^{i+j} &= \sum_{0 \leq i, j \leq n} 2^i 2^j \\ &= \left(\sum_{i=0}^n 2^i \right) \left(\sum_{j=0}^n 2^j \right) \\ &= \left(\sum_{i=0}^n 2^i \right)^2 \\ &= \left(\frac{1 - 2^{n+1}}{1 - 2} \right)^2 \\ &= (2^{n+1} - 1)^2.\end{aligned}$$

3.

$$\begin{aligned}\sum_{0 \leq i \leq j \leq n} 2^{i+j} &= \sum_{j=0}^n 2^j \sum_{i=0}^j 2^i \\ &= \sum_{j=0}^n 2^j \frac{1 - 2^{j+1}}{1 - 2} \\ &= \sum_{j=0}^n 2^j (2^{j+1} - 1) \\ &= \sum_{j=0}^n 2^{2j+1} - 2^j \\ &= 2 \sum_{j=0}^n 4^j - \sum_{j=0}^n 2^j \\ &= 2 \frac{1 - 4^{n+1}}{1 - 4} - \frac{1 - 2^{n+1}}{1 - 2} \\ &= \frac{2}{3} (4^{n+1} - 1) - 2^{n+1} + 1 \\ &= \frac{2}{3} (2^{n+1} - 1)(2^{n+1} + 1) - 2^{n+1} + 1 \\ &= (2^{n+1} - 1) \left(\frac{2}{3} 2^{n+1} - \frac{1}{3} \right) \\ &= \frac{1}{3} (2^{n+1} - 1)(2^{n+2} - 1).\end{aligned}$$

4.

$$\begin{aligned}\sum_{0 \leq i \leq j \leq n} \frac{i}{j+1} &= \sum_{j=0}^n \sum_{i=0}^j \frac{i}{j+1} \\ &= \sum_{j=0}^n \frac{1}{j+1} \sum_{i=0}^j i \\ &= \sum_{j=0}^n \frac{1}{j+1} \frac{j(j+1)}{2} \\ &= \frac{1}{2} \sum_{j=0}^n j \\ &= \frac{n(n+1)}{4}.\end{aligned}$$

5.

$$\begin{aligned}\sum_{i=1}^n \sum_{j=i}^n \frac{1}{j} &= \sum_{j=1}^n \sum_{i=1}^j \frac{1}{j} \\ &= \sum_{j=1}^n j \times \frac{1}{j} \\ &= \sum_{j=1}^n 1 \\ &= n.\end{aligned}$$

Exercice 9

Soit $n \in \mathbb{N}^*$. On a

$$\begin{aligned}\sum_{1 \leq i \leq j \leq n} ij &= \sum_{j=1}^n \sum_{i=1}^j ij \\ &= \sum_{j=1}^n j \sum_{i=1}^j i \\ &= \sum_{j=1}^n \frac{j^2(j+1)}{2} \\ &= \frac{1}{2} \sum_{j=1}^n j^3 + j^2 \\ &= \frac{1}{2} \left(\sum_{j=1}^n j^3 + \sum_{j=1}^n j^2 \right) \\ &= \frac{1}{2} \left(\frac{n^2(n+1)^2}{4} + \frac{n(n+1)(2n+1)}{6} \right) \\ &= \frac{1}{2} \left(\frac{3n^2(n+1)^2 + 2n(n+1)(2n+1)}{12} \right) \\ &= \frac{n(n+1)}{24} (3n(n+1) + 2(2n+1)) \\ &= \frac{n(n+1)}{24} (3n^2 + 7n + 2) \\ &= \frac{n(n+1)(n+2)(3n+1)}{24}.\end{aligned}$$

Exercice 10

1. On a $\max(k, j) = \begin{cases} k & \text{si } k \geq j \\ j & \text{si } j \geq k \end{cases}$ d'où

$$\begin{aligned}\sum_{k=1}^n \sum_{j=1}^n \max(k, j) &= \sum_{k=1}^n \left(\sum_{j=1}^{k-1} k + \sum_{j=k}^n j \right) \\ &= \sum_{k=1}^n \sum_{j=1}^{k-1} k + \sum_{k=1}^n \sum_{j=k}^n j \\ &= \sum_{k=1}^n k(k-1) + \sum_{j=1}^n \sum_{k=1}^j j \\ &= \sum_{k=1}^n k^2 - \sum_{k=1}^n k + \sum_{j=1}^n j^2 \\ &= \frac{n(n+1)(2n+1)}{3} - \frac{n(n+1)}{2} \\ &= \frac{n(n+1)(2(2n+1) - 3)}{6} \\ &= \frac{n(n+1)(4n-1)}{6}.\end{aligned}$$

De même, on a $\min(k, j) = \begin{cases} k & \text{si } k \leq j \\ j & \text{si } j \leq k \end{cases}$ d'où

$$\begin{aligned}
 \sum_{k=1}^n \sum_{j=1}^n \min(k, j) &= \sum_{k=1}^n \left(\sum_{j=1}^{k-1} j + \sum_{j=k}^n k \right) \\
 &= \sum_{k=1}^n \sum_{j=1}^{k-1} j + \sum_{k=1}^n \sum_{j=k}^n k \\
 &= \sum_{k=1}^n \frac{(k-1)k}{2} + \sum_{j=1}^n \sum_{k=1}^j k \\
 &= \frac{1}{2} \sum_{k=1}^n k^2 - \frac{1}{2} \sum_{k=1}^n k + \sum_{j=1}^n \frac{j(j+1)}{2} \\
 &= \frac{1}{2} \sum_{k=1}^n k^2 - \frac{1}{2} \sum_{k=1}^n k + \frac{1}{2} \sum_{j=1}^n j^2 + \frac{1}{2} \sum_{j=1}^n j \\
 &= \sum_{k=1}^n k^2 \\
 &= \frac{n(n+1)(2n+1)}{6}.
 \end{aligned}$$

2. En utilisant plusieurs fois la linéarité de la somme, on a

$$\begin{aligned}
 \sum_{k=1}^n \sum_{j=1}^n |k-j| &= \sum_{k=1}^n \left(\sum_{j=1}^n \max(k, j) - \min(k, j) \right) \\
 &= \sum_{k=1}^n \left(\sum_{j=1}^n \max(k, j) - \sum_{j=1}^n \min(k, j) \right) \\
 &= \sum_{k=1}^n \sum_{j=1}^n \max(k, j) - \sum_{k=1}^n \sum_{j=1}^n \min(k, j) \\
 &= \frac{n(n+1)(4n-1)}{6} - \frac{n(n+1)(2n+1)}{6} \\
 &= \frac{n(n+1)(4n-1-(2n+1))}{6} \\
 &= \frac{n(n+1)(2n-2)}{6} \\
 &= \frac{(n-1)n(n+1)}{3}.
 \end{aligned}$$

Exercice 11

Soit $n \in \mathbb{N}^*$. On a

$$\sum_{k=1}^n k k! = \sum_{k=1}^n (k+1-1)k! = \sum_{k=1}^n (k+1)k! - k! = \sum_{k=1}^n (k+1)! - k! = (n+1)! - 1.$$

Exercice 12

Soit $n \in \mathbb{N}^*$. On a

$$\prod_{k=1}^n (2k-1) = \frac{\prod_{k=1}^{2n} k}{\prod_{k=1}^n 2k} = \frac{(2n)!}{2^n \prod_{k=1}^n k} = \frac{(2n)!}{2^n n!}.$$

Exercice 13

1. Il s'agit d'une décomposition en éléments simples.

Supposons qu'il existe trois réels $(a, b, c) \in \mathbb{R}^3$ tels que pour tout $k \in \mathbb{R} \setminus \{0, -1, -2\}$,

$$\frac{1}{k(k+1)(k+2)} = \frac{a}{k} + \frac{b}{k+1} + \frac{c}{k+2}. \quad (1)$$

En multipliant (1) par k , on trouve $\frac{1}{(k+1)(k+2)} = a + \frac{bk}{k+1} + \frac{ck}{k+2}$. En évaluant en $k=0$, on trouve $a = \frac{1}{2}$.

De plus, en faisant tendre k vers $+\infty$, on trouve $a+b+c=0$.

En multipliant (1) par $k+1$, on trouve $\frac{1}{k(k+2)} = \frac{a(k+1)}{k} + b + \frac{c(k+1)}{k+2}$. En évaluant en $k=-1$, on trouve $b=-1$.

Enfin, puisque $c = -b - a$, on obtient $c = \frac{1}{2}$.

Ainsi, pour tout $k \in \mathbb{R} \setminus \{0, -1, -2\}$, $\frac{1}{k(k+1)(k+2)} = \frac{1}{2k} - \frac{1}{k+1} + \frac{1}{2(k+2)}$.

2. Soit $n \in \mathbb{N}^*$.

On a

$$u_n = \sum_{k=1}^n \frac{1}{k(k+1)(k+2)} = \sum_{k=1}^n \left(\frac{1}{2k} - \frac{1}{k+1} + \frac{1}{2(k+2)} \right) = \frac{1}{2} \sum_{k=1}^n \frac{1}{k} - \frac{1}{k+1} + \frac{1}{2} \sum_{k=1}^n \frac{1}{k+2} - \frac{1}{k+1}.$$

On reconnaît deux sommes télescopiques et on obtient pour tout $n \in \mathbb{N}^*$,

$$u_n = \frac{1}{2} \left(1 - \frac{1}{n+1} \right) + \frac{1}{2} \left(\frac{1}{n+2} - \frac{1}{2} \right) = -\frac{1}{2(n+1)(n+2)} + \frac{1}{4}.$$

3. On a $\lim_{n \rightarrow +\infty} -\frac{1}{2(n+1)(n+2)} = 0$ donc $\lim_{n \rightarrow +\infty} u_n = \frac{1}{4}$.

Exercice 14

Soient $n \in \mathbb{N}^*$ et $q \in \mathbb{R} \setminus \{1\}$.

1. (a) On a

$$\begin{aligned}
T_n &= \sum_{k=1}^n kq^{k-1} \\
&= \sum_{k=1}^n \sum_{i=1}^k q^{k-1} \\
&= \sum_{i=1}^n \sum_{k=i}^n q^{k-1} \\
&= \sum_{i=1}^n \sum_{j=i-1}^{n-1} q^j \quad (j = k - 1) \\
&= \sum_{i=1}^n q^{i-1} \frac{1 - q^{n-i+1}}{1 - q} \\
&= \frac{1}{1 - q} \sum_{i=1}^n (q^{i-1} - q^n) \\
&= \frac{1}{1 - q} \left(\sum_{i=1}^n q^{i-1} - \sum_{i=1}^n q^n \right) \\
&= \frac{1}{1 - q} \left(\sum_{k=0}^{n-1} q^k - nq^n \right) \quad (k = i - 1) \\
&= \frac{1}{1 - q} \left(\frac{1 - q^n}{1 - q} - nq^n \right) \\
&= \frac{1 - q^n - nq^n(1 - q)}{(1 - q)^2} \\
&= \frac{1 - (n + 1)q^n + nq^{n+1}}{(1 - q)^2}.
\end{aligned}$$

(b) S_n est une fonction dérivable sur \mathbb{R} et on a pour tout $x \in \mathbb{R}$, $S'_n(x) = \sum_{k=1}^n kx^{k-1}$.

D'autre part, pour tout $x \neq 1$, on a $S_n(x) = \frac{1 - x^{n+1}}{1 - x}$ donc pour tout $x \neq 1$, on a

$$S'_n(x) = \frac{-(n + 1)x^n(1 - x) + 1 - x^{n+1}}{(1 - x)^2} = \frac{1 - (n + 1)x^n + nx^{n+1}}{(1 - x)^2}.$$

Puisque $T_n = S'_n(q)$, on retrouve bien

$$T_n = \sum_{k=1}^n kq^{k-1} = \frac{1 - (n + 1)q^n + nq^{n+1}}{(1 - q)^2}.$$

(c) La fonction S'_n est dérivable sur \mathbb{R} et on a pour tout $x \in \mathbb{R}$ d'une part,

$$S''_n(x) = \sum_{k=2}^n k(k - 1)x^{k-2}.$$

D'autre part, en dérivant l'autre expression de S'_n , on a pour tout $x \neq 1$,

$$\begin{aligned}
S''_n(x) &= \frac{(-n(n+1)x^{n-1} + n(n+1)x^n)(1-x)^2 + 2(1-x)(1 - (n+1)x^n + nx^{n+1})}{(1-x)^4} \\
&= \frac{(-n(n+1)x^{n-1} + n(n+1)x^n)(1-x) + 2 - 2(n+1)x^n + 2nx^{n+1}}{(1-x)^3} \\
&= \frac{2 - n(n+1)x^{n-1} + 2(n-1)(n+1)x^n + n(1-n)x^{n+1}}{(1-x)^3}.
\end{aligned}$$

Finalement, on a

$$\begin{aligned}
\sum_{k=0}^n k^2 q^k &= q^2 \sum_{k=2}^n k(k-1)q^{k-2} + q \sum_{k=1}^n kq^{k-1} \\
&= q^2 S''_n(q) + qT_n \\
&= q^2 \frac{2 - n(n+1)q^{n-1} + 2(n-1)(n+1)q^n + n(1-n)q^{n+1}}{(1-q)^3} + q \frac{1 - (n+1)q^n + nq^{n+1}}{(1-q)^2}
\end{aligned}$$