Fiche n° 18. Trigonométrie et nombres complexes

Réponses

18.1 a) $\boxed{\dfrac{1}{4}\cos(3x) + \dfrac{3}{4}\cos(x)}$

18.1 b) $\boxed{-\dfrac{1}{4}\cos(4x) + \dfrac{1}{2}\cos(2x) - \dfrac{1}{4}}$

18.1 c) $\boxed{-\dfrac{1}{8}\cos(6x) + \dfrac{1}{4}\cos(4x) - \dfrac{3}{8}\cos(2x) + \dfrac{1}{4}}$

18.1 d) $\boxed{-\dfrac{\sin(9x)}{8} + \dfrac{3\sin(5x)}{8} - \dfrac{\sin(3x)}{8} - \dfrac{3\sin(x)}{8}}$

18.1 e) $\boxed{\dfrac{\cos(9x)}{8} + \dfrac{3\cos(5x)}{8} + \dfrac{\cos(3x)}{8} + \dfrac{3\cos(x)}{8}}$

18.1 f) $\boxed{-\dfrac{1}{4}\sin(11x) + \dfrac{1}{4}\sin(5x) + \dfrac{1}{2}\sin(3x)}$

18.2 a) $\boxed{2\cos\left(\dfrac{\pi}{12}\right)e^{i\frac{\pi}{12}}}$

18.2 b) $\boxed{\left(-2\cos\left(\dfrac{7\pi}{12}\right)\right)e^{-i\frac{5\pi}{12}}}$

18.2 c) $\boxed{2\sin\left(\dfrac{\pi}{12}\right)e^{\frac{-7i\pi}{12}}}$

18.2 d) $\boxed{2\cos\left(\dfrac{5\pi}{12}\right)e^{\frac{5i\pi}{12}}}$

18.2 e) $\boxed{2\cos\left(\dfrac{\pi}{12}\right)e^{i\frac{13\pi}{12}}}$

18.2 f) $\boxed{2\sin\left(\dfrac{\pi}{24}\right)e^{-i\frac{11\pi}{24}}}$

18.2 g) $\boxed{\dfrac{\cos\left(\frac{\pi}{12}\right)}{\sin\left(\frac{\pi}{24}\right)}e^{\frac{13i\pi}{24}}}$

18.2 h) $\boxed{2^{27}\cos^{27}\left(\dfrac{\pi}{12}\right)e^{i\frac{\pi}{4}}}$

18.3 a) $\boxed{2\cos\left(\dfrac{\pi}{12}\right)e^{i\frac{5\pi}{12}}}$

18.3 b) $\boxed{2\sin\left(\dfrac{\pi}{12}\right)e^{i\frac{11\pi}{12}}}$

18.4 a) $\boxed{4\cos^3(x) - 3\cos(x)}$

18.4 b) $\boxed{4\cos^3(x)\sin(x) - 4\cos(x)\sin^3(x)}$

18.5 a) $\boxed{2\cos(2x)\cos(x)}$

18.5 b) $\boxed{2\cos(4x)\sin(x)}$

18.5 c) $\boxed{2\sin(x)\sin(2x)}$

18.5 d) $\boxed{2\sin(4x)\cos(x)}$

18.6 a) $\boxed{\dfrac{\sin\left(\frac{3x}{2}\right)\sin(2x)}{\sin\left(\frac{x}{2}\right)}}$

18.6 b) $\boxed{\dfrac{\sin(8x)}{2\sin(x)}}$

18.6 c) $\boxed{0}$

Corrigés

18.1 a) On calcule :

$$\cos^3(x) = \left(\frac{e^{ix}+e^{-ix}}{2}\right)^3 = \frac{1}{8}\left(e^{3ix} + 3e^{2ix}e^{-ix} + 3e^{ix}e^{-2ix} + e^{-3ix}\right) = \frac{1}{8}\left(e^{3ix}+e^{-3ix}\right) + \frac{3}{8}\left(e^{ix}+e^{-ix}\right)$$
$$= \frac{1}{4}\cos(3x) + \frac{3}{4}\cos x.$$

18.1 b) On calcule :

$$\cos(2x)\sin^2(x) = \left(\frac{e^{2ix}+e^{-2ix}}{2}\right)\left(\frac{e^{ix}-e^{-ix}}{2i}\right)^2 = -\frac{1}{8}\left(e^{2ix}+e^{-2ix}\right)\left(e^{2ix}-2+e^{-2ix}\right)$$
$$= -\frac{1}{8}\left(e^{4ix}+e^{-4ix} - 2\left(e^{2ix}+e^{-2ix}\right)+2\right) = -\frac{1}{4}\cos(4x) + \frac{1}{2}\cos(2x) - \frac{1}{4}.$$

18.1 d) On calcule :

$$\begin{aligned}\cos(3x)\sin^3(2x) &= \left(\frac{e^{3ix}+e^{-3ix}}{2}\right)\left(\frac{e^{2ix}-e^{-2ix}}{2i}\right)^3 = -\frac{1}{16i}\left(e^{3ix}+e^{-3ix}\right)\left(e^{6ix}-3e^{2ix}+3e^{-2ix}-e^{-6ix}\right)\\ &= -\frac{1}{16i}\left(e^{9ix}-e^{-9ix}-3(e^{5ix}-e^{-5ix})+e^{3ix}-e^{-3ix}+3(e^{ix}-e^{-ix})\right)\\ &= -\frac{1}{8}\sin(9x)+\frac{3}{8}\sin(5x)-\frac{1}{8}\sin(3x)-\frac{3}{8}\sin(x).\end{aligned}$$

18.2 a) $1+e^{i\frac{\pi}{6}} = e^{i\frac{\pi}{12}}\left(e^{-\frac{i\pi}{12}}+e^{\frac{i\pi}{12}}\right) = \underbrace{2\cos\left(\frac{\pi}{12}\right)}_{>0} e^{i\frac{\pi}{12}}$.

18.2 b) $1+e^{i\frac{7\pi}{6}} = e^{\frac{7i\pi}{12}}\left(e^{-\frac{7i\pi}{12}}+e^{\frac{7i\pi}{12}}\right) = \underbrace{2\cos\left(\frac{7\pi}{12}\right)}_{<0} e^{\frac{7i\pi}{12}} = \left(-2\cos\left(\frac{7\pi}{12}\right)\right)e^{\frac{7i\pi}{12}}e^{-i\pi} = \left(-2\cos\left(\frac{7\pi}{12}\right)\right)e^{-i\frac{5\pi}{12}}$.

18.2 c) $e^{-i\frac{\pi}{6}}-1 = e^{-i\frac{\pi}{12}}\left(e^{-i\frac{\pi}{12}}-e^{i\frac{\pi}{12}}\right) = e^{-i\frac{\pi}{12}}\left(-2i\sin\left(\frac{\pi}{12}\right)\right) = 2\sin\left(\frac{\pi}{12}\right)e^{-i\frac{\pi}{12}-i\frac{\pi}{2}} = 2\sin\left(\frac{\pi}{12}\right)e^{-\frac{7i\pi}{12}}$.

18.2 d) $1+ie^{i\frac{\pi}{3}} = 1+e^{i\frac{5\pi}{6}} = e^{\frac{5i\pi}{12}}2\cos\left(\frac{5\pi}{12}\right) = 2\cos\left(\frac{5\pi}{12}\right)e^{\frac{5i\pi}{12}}$

18.2 e) $-1-e^{i\frac{\pi}{6}} = -e^{i\frac{\pi}{12}}\left(e^{-i\frac{\pi}{12}}+e^{i\frac{\pi}{12}}\right) = \underbrace{-2\cos\left(\frac{\pi}{12}\right)}_{<0} e^{i\frac{\pi}{12}} = 2\cos\left(\frac{\pi}{12}\right)e^{i\frac{\pi}{12}+i\pi} = 2\cos\left(\frac{\pi}{12}\right)e^{i\frac{13\pi}{12}}$.

18.2 f) $1-e^{i\frac{\pi}{12}} = e^{i\frac{\pi}{24}}\left(-2i\sin\left(\frac{\pi}{24}\right)\right) = 2\sin\left(\frac{\pi}{24}\right)e^{i\frac{\pi}{24}}e^{-i\frac{\pi}{2}} = 2\sin\left(\frac{\pi}{24}\right)e^{-i\frac{11\pi}{24}}$.

18.2 g) On fait le quotient de a) et f).

18.2 h) $\left(1+e^{i\frac{\pi}{6}}\right)^{27} = \left(2\cos\left(\frac{\pi}{12}\right)e^{i\frac{\pi}{12}}\right)^{27} = 2^{27}\cos^{27}\left(\frac{\pi}{12}\right)e^{i\frac{\pi}{4}}$.

18.3 a) $e^{i\frac{\pi}{3}}+e^{i\frac{\pi}{2}} = e^{i\frac{\frac{\pi}{3}+\frac{\pi}{2}}{2}}\left(e^{i\frac{\frac{\pi}{3}-\frac{\pi}{2}}{2}}+e^{i\frac{\frac{\pi}{2}-\frac{\pi}{3}}{2}}\right) = \underbrace{2\cos\left(\frac{\pi}{12}\right)}_{>0} e^{i\frac{5\pi}{12}}$.

18.3 b) $e^{i\frac{\pi}{3}}-e^{i\frac{\pi}{2}} = e^{i\frac{\frac{\pi}{3}+\frac{\pi}{2}}{2}}\left(e^{i\frac{\frac{\pi}{3}-\frac{\pi}{2}}{2}}-e^{i\frac{\frac{\pi}{2}-\frac{\pi}{3}}{2}}\right) = 2\sin\left(\frac{\pi}{12}\right)ie^{5i\frac{\pi}{12}} = 2\sin\left(\frac{\pi}{12}\right)e^{5i\frac{\pi}{12}+i\frac{\pi}{2}} = \underbrace{2\sin\left(\frac{\pi}{12}\right)}_{>0} e^{i\frac{11\pi}{12}}$.

18.4 a) On calcule :

$$\begin{aligned}\cos(3x) &= \mathrm{Re}(e^{3ix}) = \mathrm{Re}\left((e^{ix})^3\right) = \mathrm{Re}\left((\cos(x)+i\sin(x))^3\right)\\ &= \mathrm{Re}(\cos^3(x)+3i\cos^2(x)\sin(x)-3\cos(x)\sin^2(x)-i\sin^3(x))\\ &= \cos^3(x)-3\cos(x)\sin^2(x) = \cos^3(x)-3\cos(x)(1-\cos^2(x))\\ &= 4\cos^3(x)-3\cos(x).\end{aligned}$$

18.4 b) On calcule :

$$\begin{aligned}\sin(4x) &= \mathrm{Im}(e^{4ix}) = \mathrm{Im}\left((e^{ix})^4\right) = \mathrm{Im}\left((\cos(x)+i\sin(x))^4\right)\\ &= \mathrm{Im}(\cos^4(x)+4i\cos^3(x)\sin(x)-6\cos^2(x)\sin^2(x)-4i\cos(x)\sin^3(x)+\sin^4(x))\\ &= 4\cos^3(x)\sin(x)-4\cos(x)\sin^3(x).\end{aligned}$$

18.5 a) $\cos(x)+\cos(3x) = \mathrm{Re}(e^{ix}+e^{3ix}) = \mathrm{Re}\left(e^{i\frac{x+3x}{2}}(e^{i(-x)}+e^{ix})\right) = \mathrm{Re}\left(e^{2ix}2\cos(x)\right) = 2\cos(2x)\cos(x)$.

18.5 b) $\sin(5x) - \sin(3x) = \mathrm{Im}(e^{5ix} - e^{3ix}) = \mathrm{Im}\big(e^{4ix}(e^{ix} - e^{-ix})\big) = \mathrm{Im}\big(e^{4ix}2i\sin(x)\big) = 2\cos(4x)\sin(x)$.

18.5 c) $\cos(x) - \cos(3x) = \mathrm{Re}(e^{ix} - e^{3ix}) = \mathrm{Re}\Big(e^{i\frac{x+3x}{2}}(e^{i(-x)} - e^{ix})\Big) = \mathrm{Re}\big(e^{2ix}(-2i)\sin(x)\big) = 2\sin(x)\sin(2x)$.

18.5 d) $\sin(3x) + \sin(5x) = \mathrm{Im}(e^{3ix} + e^{5ix}) = \mathrm{Im}\big(e^{4ix}(e^{-ix} + e^{ix})\big) = \mathrm{Im}\big(e^{4ix}2\cos(x)\big) = 2\sin(4x)\cos(x)$.

18.6 a) Si $x \in 2\pi\mathbb{Z}$, alors cette somme vaut 0. Sinon, $\sin(x) + \sin(2x) + \sin(3x) = \mathrm{Im}(e^{ix} + e^{2ix} + e^{3ix})$
$= \mathrm{Im}\big(1 + e^{ix} + (e^{ix})^2 + (e^{ix})^3\big)$. Or, $e^{ix} \neq 1$ donc $1 + e^{ix} + (e^{ix})^2 + (e^{ix})^3 = \dfrac{1 - e^{4ix}}{1 - e^{ix}}$.

On utilise maintenant l'astuce de l'arc moitié. On obtient,

$$\sin(x) + \sin(2x) + \sin(3x) = \mathrm{Im}\left(\frac{e^{2ix}}{e^{i\frac{x}{2}}}\frac{-2i\sin(2x)}{-2i\sin\left(\frac{x}{2}\right)}\right) = \mathrm{Im}\left(e^{i\frac{3x}{2}}\frac{\sin(2x)}{\sin\left(\frac{x}{2}\right)}\right) = \frac{\sin\left(\frac{3x}{2}\right)\sin(2x)}{\sin\left(\frac{x}{2}\right)}.$$

18.6 b) Si $x \in 2\pi\mathbb{Z}$, alors cette somme vaut 4.

Si x est de la forme $\pi + 2k\pi$ avec $k \in \mathbb{Z}$, la somme vaut -4.

Sinon, on calcule :

$$\begin{aligned}\cos(x) + \cos(3x) + \cos(5x) + \cos(7x) &= \mathrm{Re}(e^{ix} + e^{3ix} + e^{5ix} + e^{7ix}) \\ &= \mathrm{Re}\Big(e^{ix}\big(1 + (e^{2ix}) + (e^{2ix})^2 + (e^{2ix})^3\big)\Big).\end{aligned}$$

Or, $e^{2ix} \neq 1$ donc

$$e^{ix}\big(1 + (e^{2ix}) + (e^{2ix})^2 + (e^{2ix})^3\big) = e^{ix}\frac{1 - (e^{2ix})^4}{1 - e^{2ix}} = e^{ix}\frac{1 - (e^{8ix})}{1 - e^{2ix}} = e^{ix}\frac{e^{4ix}}{e^{ix}}\frac{-2i\sin(4x)}{-2i\sin(x)} = e^{4ix}\frac{\sin(4x)}{\sin(x)}.$$

Finalement, on a

$$\cos(x) + \cos(3x) + \cos(5x) + \cos(7x) = \frac{\cos(4x)\sin(4x)}{\sin(x)} = \frac{\sin(8x)}{2\sin(x)}.$$

18.6 c) On calcule :

$$\cos(x) + \cos\left(x + \frac{2\pi}{3}\right) + \cos\left(x + \frac{4\pi}{3}\right) = \mathrm{Re}\left(e^{ix} + e^{i(x+\frac{2\pi}{3})} + e^{i(x+\frac{4\pi}{3})}\right) = \mathrm{Re}\left(e^{ix}\underbrace{(1 + j + j^2)}_{=0}\right) = 0.$$
