Fiche n° 16. Systèmes linéaires

Réponses

16.1 a) .... $\{(3,1)\}$

16.1 b) .... $\{(7,2)\}$

16.1 c) .... $\left\{\left(\frac{1}{3},\frac{2}{3}\right)\right\}$

16.1 d) .... $\left\{\left(\frac{\sqrt{2}}{3},\frac{\sqrt{2}}{2}\right)\right\}$

16.2 a) .... $\left\{\left(1-\frac{a}{4},\frac{-1}{2}+\frac{3}{8}a\right)\right\}$

16.2 b) .... $(2,-3)$

16.2 c) .... $\left\{\left(\frac{1}{13}a+\frac{5}{13}a^2,\frac{2}{13}a-\frac{3}{13}a^2\right)\right\}$

16.2 d) .... $(a-2a^2,a+a^2)$

16.3 a) .... $\{(1+z,-z,z);z\in\mathbb{R}\}$

16.3 b) .... $\{(1,y,3+2y);y\in\mathbb{R}\}$

16.3 c) .... $\left\{\left(\frac{13}{6}-\frac{5}{3}z,-\frac{1}{3}+\frac{4}{3}z,z\right);z\in\mathbb{R}\right\}$

16.3 d) .... $\left\{\left(x,\frac{-5}{12}-\frac{3}{2}x,\frac{-25}{24}-\frac{7}{4}x\right);x\in\mathbb{R}\right\}$

16.4 a) .... $\{(2,-1,3)\}$

16.4 b) .... $\{(-1,4,2)\}$

16.4 c) .... $\varnothing$

16.4 d) .... $\left\{\left(-\frac{2}{7}-z,\frac{-3}{7},z\right);z\in\mathbb{R}\right\}$

16.5 a) .... $\left\{\left(1,\frac{1}{2},\frac{1}{2}\right)\right\}$

16.5 b) .... $\varnothing$

16.5 c) .... $\{(5z,1-4z,z)\ ;\ z\in\mathbb{R}\}$

16.5 d) .... $\left\{\left(1,\frac{1}{a+2},\frac{1}{a+2}\right)\right\}$

16.6 a) .... $\{(5,3,-1)\}$

16.6 b) .... $\varnothing$

16.6 c) .... $\left\{\left(\frac{a^2+a-1}{a^3-1}c,\frac{a^2-a-1}{a^3-1}c,\frac{-a^2+a+1}{a^3-1}c\right)\right\}$

16.7 a) .... $\{(0,0,0)\}$

16.7 b) .... $\{(x,y,-x-y)\ ;\ (x,y)\in\mathbb{R}^2\}$

16.7 c) .... $\{(z,z,z)\ ;\ z\in\mathbb{R}\}$

Corrigés

16.1 a)

$$\begin{cases} x-2y=1 \\ 3x+4y=13 \end{cases} \underset{L_2\leftarrow L_2-3L_1}{\Leftrightarrow} \begin{cases} x-2y=1 \\ 10y=10 \end{cases} \Leftrightarrow \begin{cases} x=1+2\times 1 \\ y=1 \end{cases} \Leftrightarrow \begin{cases} y=1 \\ x=3 \end{cases}$$

16.1 b) $\begin{cases} 2x+y=16 \\ x-y=5 \end{cases} \underset{L_1\leftarrow L_1+L_2}{\Leftrightarrow} \begin{cases} 3x=21 \\ x-y=5 \end{cases} \Leftrightarrow \begin{cases} x=7 \\ y=7-5=2 \end{cases}$

16.1 c) $\begin{cases} 3x-6y=-3 \\ 2x+2y=2 \end{cases} \Leftrightarrow \begin{cases} x-2y=-1 \\ x+\ y=1 \end{cases} \underset{L_2\leftarrow L_2-L_1}{\Leftrightarrow} \begin{cases} x-2y=-1 \\ 3y=2 \end{cases} \Leftrightarrow \begin{cases} x=-1+2\times\frac{2}{3}=\frac{1}{3} \\ y=\frac{2}{3} \end{cases}$

16.1 d)

$$\begin{cases} 3x-4y=-\sqrt{2} \\ 6x+2y=3\sqrt{2} \end{cases} \underset{L_2\leftarrow L_2-2L_1}{\Leftrightarrow} \begin{cases} 3x-4y=-\sqrt{2} \\ 10y=5\sqrt{2} \end{cases} \Leftrightarrow \begin{cases} 3x=4\times\frac{\sqrt{2}}{2}-\sqrt{2} \\ y=\frac{\sqrt{2}}{2} \end{cases} \Leftrightarrow \begin{cases} x=\frac{\sqrt{2}}{3} \\ y=\frac{\sqrt{2}}{2} \end{cases}$$

16.2 a) $\left\{\begin{array}{l}3x+2y=2\\2x+4y=a\end{array}\right. \underset{L_2\leftarrow L_2-2L_1}{\Leftrightarrow} \left\{\begin{array}{l}3x+2y=2\\-4x=a-4\end{array}\right. \Leftrightarrow \left\{\begin{array}{l}y=1-\dfrac{3}{2}x\\x=1-\dfrac{a}{4}\end{array}\right. \Leftrightarrow \left\{\begin{array}{l}x=1-\dfrac{a}{4}\\y=1-\dfrac{3}{2}+\dfrac{3}{8}a=\dfrac{-1}{2}+\dfrac{3}{8}a\end{array}\right.$

16.2 b)

$$\left\{\begin{array}{l}x-ay=3a+2\\ax+y=2a-3\end{array}\right. \underset{L_2\leftarrow L_2-aL_1}{\Leftrightarrow} \left\{\begin{array}{l}x\ -\ ay=3a+2\\(a^2+1)y=-3-3a^2\end{array}\right. \underset{1+a^2\neq 0}{\Longleftrightarrow} \left\{\begin{array}{l}x=-3a+3a+2=2\\y=-3\end{array}\right.$$

16.2 c)

$$\left\{\begin{array}{l}3x+5y=a\\2x-y=a^2\end{array}\right. \underset{L_1\leftarrow 5L_2+L_1}{\Leftrightarrow} \left\{\begin{array}{l}13x\quad=a+5a^2\\2x-y=a^2\end{array}\right. \Leftrightarrow \left\{\begin{array}{l}x=\dfrac{1}{13}a+\dfrac{5}{13}a^2\\y=2x-a^2\end{array}\right. \Leftrightarrow \left\{\begin{array}{l}x=\dfrac{1}{13}a+\dfrac{5}{13}a^2\\y=2\times\left(\dfrac{1}{13}a+\dfrac{5}{13}a^2\right)-a^2=\dfrac{2}{13}a-\dfrac{3}{13}a^2\end{array}\right.$$

16.2 d)

$$\left\{\begin{array}{l}x+2y=3a\\2x+3y=5a-a^2\end{array}\right. \underset{L_2\leftarrow L_2-2L_1}{\Longleftrightarrow} \left\{\begin{array}{l}x+2y=3a\\-y=-a^2-a\end{array}\right. \Leftrightarrow \left\{\begin{array}{l}x=3a-2(a+a^2)=a-2a^2\\y=a+a^2\end{array}\right.$$

16.3 a) $\left\{\begin{array}{l}x+2y+z=1\\3x+y-2z=3\end{array}\right. \underset{L_2\leftarrow L_2-3L_1}{\Longleftrightarrow} \left\{\begin{array}{l}x+2y+z=1\\-5y-5z=0\end{array}\right. \Leftrightarrow \left\{\begin{array}{l}x+2y=1-z\\y=-z\end{array}\right. \Leftrightarrow \left\{\begin{array}{l}x=1+z\\y=-z\end{array}\right.$

16.3 b) $\left\{\begin{array}{l}3x-2y+z=6\\x+2y-z=-2\end{array}\right. \underset{L_1\leftarrow L_1+L_2}{\Longleftrightarrow} \left\{\begin{array}{l}4x=4\\x+2y-z=-2\end{array}\right. \Leftrightarrow \left\{\begin{array}{l}x=1\\2y-z=-3\end{array}\right. \Leftrightarrow \left\{\begin{array}{l}x=1\\z=2y+3\end{array}\right.$

16.3 c)

$$\left\{\begin{array}{l}x-y+3z=\dfrac{5}{2}\\x+2y-z=\dfrac{3}{2}\end{array}\right. \underset{L_2\leftarrow L_2-L_1}{\Longleftrightarrow} \left\{\begin{array}{l}x-y+3z=\dfrac{5}{2}\\3y-4z=-1\end{array}\right. \Leftrightarrow \left\{\begin{array}{l}x=\dfrac{-1}{3}+\dfrac{4}{3}z-3z+\dfrac{5}{2}\\y=\dfrac{-1}{3}+\dfrac{4}{3}z\end{array}\right. \Leftrightarrow \left\{\begin{array}{l}x=\dfrac{13}{6}-\dfrac{5}{3}z\\y=\dfrac{-1}{3}+\dfrac{4}{3}z\end{array}\right.$$

16.3 d)

$$\left\{\begin{array}{l}5x+y+2z=-\dfrac{5}{2}\\2x-y+2z=-\dfrac{5}{3}\end{array}\right. \underset{L_2\leftarrow L_2+L_1}{\Longleftrightarrow} \left\{\begin{array}{l}5x+y+2z=-\dfrac{5}{2}\\7x\quad+\quad 4z=-\dfrac{25}{6}\end{array}\right. \Leftrightarrow \left\{\begin{array}{l}y+2z=-\dfrac{5}{2}-5x\\4z=-\dfrac{25}{6}-7x\end{array}\right.$$

$$\Leftrightarrow \left\{\begin{array}{l}y=-\dfrac{5}{2}-5x+\dfrac{25}{12}+\dfrac{7}{2}x=\dfrac{-5}{12}-\dfrac{3}{2}x\\z=\dfrac{-25}{24}-\dfrac{7}{4}x\end{array}\right.$$

16.4 a)

$$\left\{\begin{array}{l}x+2y-z=-3\\2x-y+z=8\\3x+y+2z=11\end{array}\right. \underset{\substack{L_2\leftarrow L_2-2L_1\\L_3\leftarrow L_3-3L_1}}{\Longleftrightarrow} \left\{\begin{array}{l}x+2y-z=-3\\-5y+3z=14\\-5y+5z=20\end{array}\right. \underset{L_3\leftarrow L_3-L_2}{\Longleftrightarrow} \left\{\begin{array}{l}x+2y-z=-3\\-5y+3z=14\\2z=6\end{array}\right.$$

$$\Longleftrightarrow \left\{\begin{array}{l}x+2y-3=-3\\-5y+3\times 3=14\\z=3\end{array}\right. \Leftrightarrow \left\{\begin{array}{l}x+2y=0\\y=-1\\z=3\end{array}\right. \Leftrightarrow \left\{\begin{array}{l}x=2\\y=-1\\z=3\end{array}\right.$$

16.4 b)

$$\begin{cases} a-b-c=-7 \\ 3a+2b-c=3 \\ 4a+b+2c=4 \end{cases} \underset{\substack{L_2 \leftarrow L_2-3L_1 \\ L_3 \leftarrow L_3-4L_1}}{\iff} \begin{cases} a-b-c=-7 \\ 5b+2c=24 \\ 5b+6c=32 \end{cases} \underset{L_3\leftarrow L_3-L_2}{\iff} \begin{cases} a-b-c=-7 \\ 5b+2c=24 \\ 4c=8 \end{cases}$$

$$\iff \begin{cases} a-b-2=-7 \\ 5b+2\times 2=24 \\ c=2 \end{cases} \iff \begin{cases} a=-5+4=-1 \\ b=4 \\ c=2 \end{cases}$$

16.4 c) $\begin{cases} x+3y+z=1 \\ 2x-y+2z=-1 \\ x+10y+z=0 \end{cases} \underset{\substack{L_2 \leftarrow L_2-2L_1 \\ L_3 \leftarrow L_3-L_1}}{\iff} \begin{cases} x+3y+z=1 \\ -7y=-3 \\ 7y=-1 \end{cases} \underset{L_3\leftarrow L_3+L_2}{\iff} \begin{cases} x+3y+z=1 \\ -7y=-3 \\ 0=-4 \end{cases}.$

Le système est incompatible.

16.4 d) On va extraire y de la deuxième équation, puis résoudre par substitution.

$$\begin{cases} 3x+2y+3z=0 \\ 2x-y+2z=-1 \\ 4x+5y+4z=1 \end{cases} \underset{\substack{L_1 \leftarrow L_1+2L_2 \\ L_3 \leftarrow L_3+5L_2}}{\iff} \begin{cases} 7x+\quad 7z=-2 \\ 2x-y+2z=-1 \\ 14x+\quad 14z=-4 \end{cases} \Leftrightarrow \begin{cases} 7x+\quad 7z=-2 \\ 2x-y+2z=-1 \\ 0=0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x=-z-\dfrac{2}{7} \\ y=2x+2z+1 \end{cases} \Leftrightarrow \begin{cases} x=-z-\dfrac{2}{7} \\ \\ y=-2z-\dfrac{4}{7}+2z+1=\dfrac{3}{7} \end{cases}$$

16.5 a)

$$\begin{cases} x+y-z=1 \\ x+2y=2 \\ 2x+2z=3 \end{cases} \underset{\substack{L_2 \leftarrow L_2-L_1 \\ L_3 \leftarrow L_3-2L_1}}{\iff} \begin{cases} x+y-z=1 \\ y+z=1 \\ -2y+4z=1 \end{cases} \underset{L_3\leftarrow L_3+2L_2}{\iff} \begin{cases} x+y-z=1 \\ y+z=1 \\ 6z=3 \end{cases} \Leftrightarrow \begin{cases} x=1 \\ \\ y=\dfrac{1}{2} \\ \\ z=\dfrac{1}{2} \end{cases}$$

16.5 b)

$$\begin{cases} x+y-z=1 \\ x+2y-2z=2 \\ 2x-2y+2z=3 \end{cases} \underset{\substack{L_2 \leftarrow L_2-L_1 \\ L_3 \leftarrow L_3-2L_1}}{\iff} \begin{cases} x+y-z=1 \\ y-z=1 \\ -4y+4z=1 \end{cases} \underset{L_3\leftarrow L_3+4L_2}{\iff} \begin{cases} x+y-z=1 \\ y-z=1 \\ 0=5 \end{cases}$$

Le système est incompatible.

16.5 c)

$$\begin{cases} x+y-z=1 \\ x+2y+3z=2 \\ 2x+3y+2z=3 \end{cases} \underset{\substack{L_2 \leftarrow L_2-L_1 \\ L_3 \leftarrow L_3-2L_1}}{\iff} \begin{cases} x+y-z=1 \\ y+4z=1 \\ y+4z=1 \end{cases} \Leftrightarrow \begin{cases} x=-(1-4z)+z+1 \;=\; 5z \\ y=1-4z \end{cases}.$$

16.5 d)

$$\begin{cases} x+y-z=1 \\ x+2y+az=2 \\ 2x+ay+2z=3 \end{cases} \underset{\substack{L_2 \leftarrow L_2 - L_1 \\ L_3 \leftarrow L_3 - 2L_1}}{\iff} \begin{cases} x+y-z=1 \\ y+(a+1)z=1 \\ (a-2)y+4z=1 \end{cases} \underset{L_3 \leftarrow L_3+(2-a)L_2}{\iff} \begin{cases} x+y-z=1 \\ y+(a+1)z=1 \\ (4+(2-a)(a+1))z=3-a \end{cases}$$

$$\Leftrightarrow \begin{cases} x+y-z=1 \\ y+(a+1)z=1 \\ (4+a+2-a^2)z=3-a \end{cases} \Leftrightarrow \begin{cases} x+y-z=1 \\ y+(a+1)z=1 \\ (-a^2+a+6)z=3-a \end{cases}$$

On factorise le trinôme $-(a^2-a-6) = -(a+2)(a-3)$ qui est non nul dans le cas étudié.

D'où : $\begin{cases} x+y-z=1 \\ y+(a+1)z=1 \\ (-a^2+a+6)z=3-a \end{cases} \Leftrightarrow \begin{cases} x=1-y+z \\ y=1-(a+1)\times\dfrac{1}{a+2} \\ z=\dfrac{3-a}{-(a+2)(a-3)}=\dfrac{1}{a+2} \end{cases} \Leftrightarrow \begin{cases} x=1-\dfrac{1}{a+2}+\dfrac{1}{a+2}=1 \\ y=\dfrac{a+2-a-1}{a+2}=\dfrac{1}{a+2} \\ z=\dfrac{1}{a+2} \end{cases}$

16.6 a)

$$\begin{cases} x - 2z=7 \\ 2x-y=7 \\ 2y-z=7 \end{cases} \underset{L_2 \leftarrow L_2-2L_1}{\iff} \begin{cases} x - 2z=7 \\ -y+4z-7 \\ 2y-z=7 \end{cases} \underset{L_3 \leftarrow L_3+2L_2}{\iff} \begin{cases} x - 2z=7 \\ -y+4z-7 \\ -7z=-7 \end{cases} \Leftrightarrow \begin{cases} x=7+2z \\ y=7+4z \\ z=-1 \end{cases} \Leftrightarrow \begin{cases} x=5 \\ y=3 \\ z=-1 \end{cases}$$

16.6 b)

$$\begin{cases} x-z=2 \\ x-y=2 \\ y-z=2 \end{cases} \underset{L_2 \leftarrow L_1-L_2}{\iff} \begin{cases} x - z=2 \\ y-z=0 \\ y-z=2 \end{cases}$$

Le système est incompatible.

16.6 c)

$$\begin{cases} x - az=c \\ ax-y=c \\ ay-z=c \end{cases} \underset{L_2 \leftarrow L_2-aL_1}{\iff} \begin{cases} x - az=c \\ -y+a^2z=(1-a)c \\ ay - z=c \end{cases} \underset{L_3 \leftarrow L_3+aL_2}{\iff} \begin{cases} x=c+az \\ y=(a-1)c+a^2z \\ (a^3-1)z=(1+a-a^2)c \end{cases}$$

a est un réel différent de 1 donc $a^3-1 \neq 0$, on peut déterminer z dans la troisième équation.

$$\begin{cases} x=c+az \\ y=(a-1)c+a^2z \\ (a^3-1)z=(1+a-a^2)c \end{cases} \Leftrightarrow \begin{cases} z=c\dfrac{-a^2+a+1}{(a-1)(a^2+a+1}=\dfrac{-a^2+a+1}{a^3-1}c \\ y=(a-1)c+a^2\dfrac{-a^2+a+1}{a^3-1}c=\dfrac{a^2-a+1}{a^3-1}c \\ x=c+a\dfrac{-a^2+a+1}{a^3-1}c=\dfrac{a^2+a-1}{a^3-1}c \end{cases}$$

16.7 a)

$$\begin{cases} 4x+y+z=x \\ x+4y+z=y \\ x+y+4z=z \end{cases} \Leftrightarrow \begin{cases} 3x+y+z=0 \\ x+3y+z=0 \\ x+y+3z=0 \end{cases} \underset{L_1 \leftarrow \frac{1}{5}(L_1+L_2+L_3)}{\iff} \begin{cases} x+y+z=0 \\ x+3y+z=0 \\ x+y+3z=0 \end{cases} \underset{\substack{L_2 \leftarrow L_2-L_1 \\ L_3 \leftarrow L_3-L_1}}{\iff} \begin{cases} x+y+z=0 \\ 2y=0 \\ 2z=0 \end{cases}$$

$$\Leftrightarrow x=y=z=0$$

16.7 b)

$$\left\{\begin{array}{l} 4x+y+z=3x \\ x+4y+z=3y \\ x+y+4z=3z \end{array}\right. \Leftrightarrow \left\{\begin{array}{l} x+y+z=0 \\ x+y+z=0 \\ x+y+z=0 \end{array}\right. \Leftrightarrow z=-x-y$$

16.7 c)

$$\left\{\begin{array}{l} 4x+y+z=6x \\ x+4y+z=6y \\ x+y+4z=6z \end{array}\right. \Leftrightarrow \left\{\begin{array}{l} -2x+y+z=0 \\ x-2y+z=0 \\ x+y-2z=0 \end{array}\right. \underset{L_1 \leftarrow L_1+L_2+L_3}{\Longleftrightarrow} \left\{\begin{array}{l} 0=0 \\ x-2y+z=0 \\ x+y-2z=0 \end{array}\right. \Leftrightarrow \left\{\begin{array}{l} x-2y+z=0 \\ 3y-3z=0 \end{array}\right. \Leftrightarrow \left\{\begin{array}{l} x=z \\ y=z \end{array}\right.$$