TD4 :

exo 18 : On a :

$$(1+\sqrt{2})^n + (1-\sqrt{2})^n = \sum_{k=0}^{n} \binom{n}{k} \sqrt{2}^k 1^{n-k} + \sum_{k=0}^{n} \binom{n}{k} (-\sqrt{2})^k 1^{n-k}$$

$$= \sum_{k=0}^{n} \binom{n}{k} \left(\sqrt{2}^k + (-1)^k \sqrt{2}^k \right)$$

$$= \sum_{k=0}^{n} \binom{n}{k} \sqrt{2}^k \left(1 + (-1)^k \right)$$

Or si k est impair, $1+(-1)^k = 0$ et si k est pair $1+(-1)^k = 2$

donc $(1+\sqrt{2})^n + (1-\sqrt{2})^n = \sum_{\substack{k=0 \\ k \text{ pair}}}^{n} \binom{n}{k} \sqrt{2}^k \times 2$

Or si k est pair, alors $\sqrt{2}^k = 2^{\left(\frac{k}{2}\right)} \in \mathbb{N}$ et $\binom{n}{k} \in \mathbb{N}$.

Ainsi $(1+\sqrt{2})^n + (1-\sqrt{2})^n \in \mathbb{N}$.