Exercice 1
Pour $n \in \mathbb{N}^*$ soit $u_n = \sum_{k=n}^{2n} \frac{1}{k}$.

1. Calculer $u_{n+1} - u_n$.
2. En déduire que (u_n) est décroissante.

Exercice 2
Calculer les sommes et produits suivants :

1. $C_n = \sum_{k=0}^{n} \binom{n}{k} \cos(k\theta)$ et $S_n = \sum_{k=0}^{n} \binom{n}{k} \sin(k\theta)$
2. $P_n = \prod_{k=\cancel{1}\,2}^{n} \frac{k^2-1}{k^2}$
3. $S_n = \sum_{k=1}^{n} k\, k!$
4. $T_n = \sum_{k=0}^{n} k^3$ via un retournement

exo 1 : 1) $u_{n+1} - u_n = \sum_{k=n+1}^{2(n+1)} \frac{1}{k} - \sum_{k=n}^{2n} \frac{1}{k} = \left(\sum_{k=n}^{2n} \frac{1}{k} + \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n}\right) - \sum_{k=n}^{2n} \frac{1}{k}$

$= \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n}$

2) $u_{n+1} - u_n = \frac{(2n+2)n + (2n+1)n - (2n+2)(2n+1)}{(2n+1)(2n+2)n} = \frac{-3n-2}{(2n+1)(2n+2)n} \leq 0$ donc $(u_n) \searrow$

exo 2 : 1) $C_n + iS_n = \sum_{k=0}^{n} \binom{n}{k} e^{ik\theta} = \sum_{k=0}^{n} \binom{n}{k} (e^{i\theta})^k 1^{n-k} = (e^{i\theta} + 1)^n$

$= (e^{i\theta/2})^n (e^{i\theta/2} + e^{-i\theta/2})^n = 2^n \cos^n\left(\frac{\theta}{2}\right) e^{i\frac{n\theta}{2}} = 2^n \cos^n\left(\frac{\theta}{2}\right)\left(\cos\left(\frac{n\theta}{2}\right) + i\sin\left(\frac{n\theta}{2}\right)\right)$

Donc $C_n = 2^n \cos^n\left(\frac{\theta}{2}\right) \cos\left(\frac{n\theta}{2}\right)$ et $S_n = 2^n \cos^n\left(\frac{\theta}{2}\right) \sin\left(\frac{n\theta}{2}\right)$

2) $P_n = \prod_{k=2}^{n} \frac{(k-1)(k+1)}{k^2} = \prod_{k=2}^{n} \frac{k-1}{k} \times \prod_{k=2}^{n} \frac{k+1}{k} = \frac{2-1}{n} \times \frac{n+1}{2} = \frac{n+1}{2n}$

3) $S_n = \sum_{k=1}^{n} (k+1-k)\, k! = \sum_{k=1}^{n} (k+1)k! - k! = \sum_{k=1}^{n} (k+1)! - k!$

$= (n+1)! - 1! = (n+1)! - 1$

4) $T_n = \sum_{j=0}^{n} (n-j)^3 = \sum_{j=0}^{n} n^3 - 3n^2 \sum_{j=0}^{n} j + 3n \sum_{j=0}^{n} j^2 - \sum_{j=0}^{n} j^3$

via $j = n-k$
$\Leftrightarrow k = n-j$
$0 \leq k \leq n \Leftrightarrow 0 \leq j \leq n$

donc $T_n = (n+1)n^3 - 3n^2 \frac{n(n+1)}{2} + 3n \frac{n(n+1)(2n+1)}{6} - T_n$

d'où (après simplification) $T_n = \frac{n^2(n+1)^2}{4}$