exo 19 :

1) $S_n = \sum_{k=0}^{n} k\binom{n}{k}$ $\qquad \binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$ si $k \in [\![1,n]\!]$

donc $k\binom{n}{k} = n\binom{n-1}{k-1}$

$$= 0\binom{n}{0} + \sum_{k=1}^{n} k\binom{n}{k}$$

$$= \sum_{k=1}^{n} n\binom{n-1}{k-1}$$

$$= n\sum_{j=0}^{n-1}\binom{n-1}{j} = n2^{n-1}$$

2) $T_n = \sum_{k=0}^{n} k^2\binom{n}{k} = \sum_{k=1}^{n} k \times \left(k\binom{n}{k}\right)$

$$= \sum_{k=1}^{n} nk\binom{n-1}{k-1} = n\sum_{k=1}^{n}\left((k-1)+1\right)\binom{n-1}{k-1}$$

$$= n\sum_{k=1}^{n}(k-1)\binom{n-1}{k-1} + n\sum_{k=1}^{n}\binom{n-1}{k-1}$$

$$= n\sum_{k=2}^{n}(k-1)\binom{n-1}{k-1} + n\sum_{j=0}^{n-1}\binom{n-1}{j}$$

$\binom{n-1}{k-1} = \frac{n-1}{k-1}\binom{n-2}{k-2}$

$$= n\sum_{k=2}^{n}(n-1)\binom{n-2}{k-2} + n2^{n-1}$$

$$= n(n-1)\sum_{j=0}^{n-2}\binom{n-2}{j} + n2^{n-1}$$

$$= n(n-1)2^{n-2} + n2^{n-1}$$

$$= \left(n(n-1) + 2n\right)2^{n-2}$$

$$= n(n+1)2^{n-2}$$