Remédiation 6

Exercice 1

1) $\displaystyle\sum_{j=0}^{n-2} 3^j = \frac{1-3^{n-1}}{1-3} = \boxed{\frac{3^{n-1}-1}{2}}$

2) $\displaystyle\sum_{k=0}^{n} \binom{n}{k} 2^{-k} = \left(\frac{1}{2}+1\right)^n = \boxed{\left(\frac{3}{2}\right)^n}$

3) $\displaystyle\sum_{k=0}^{n} \binom{n}{k} x^k y^{2k+1} = \boxed{y \times (xy^2+1)^n}$

4) $\displaystyle\sum_{k=1}^{n+1} \binom{n+1}{k} (x-1)^k = (x-1+1)^{n+1} - \binom{n+1}{0}(x-1)^0$
$= \boxed{x^{n+1} - 1}$

5) $\displaystyle\sum_{k=2}^{n+2} \binom{n+2}{k-1} = \sum_{k=1}^{n+1} \binom{n+2}{k}$
$\displaystyle= \sum_{k=0}^{n+2} \binom{n+2}{k} - \binom{n+2}{0} - \binom{n+2}{n+2}$
$= \boxed{2^{n+2} - 2}$

6) $\displaystyle\prod_{k=1}^{n} \frac{2}{k} = \boxed{\frac{2^n}{n!}}$

7) $\displaystyle\prod_{k=2}^{n-1} 3^{-k} = 3^{-\sum_{k=2}^{n-1} k}$
$= 3^{-\left(\sum_{k=1}^{n-1} k - 1\right)}$
$= \boxed{3^{-\frac{(n-1)n}{2}+1}}$

8) $\displaystyle\sum_{k=0}^{n} (1-x)^k (1+x)^{n-k} = (1+x)^n \sum_{k=0}^{n} \left(\frac{1-x}{1+x}\right)^k$
$\displaystyle= (1+x)^n \times \frac{1-\left(\frac{1-x}{1+x}\right)^{n+1}}{1-\frac{1-x}{1+x}}$
$\displaystyle= \frac{(1+x)^{n+1} - (1-x)^{n+1}}{1+x-(1-x)} = \boxed{\frac{(1+x)^{n+1} - (1-x)^{n+1}}{2x}}$

(correct si $\frac{1-x}{1+x} \neq 1$ i.e. $x \neq 0$; et vaut $n+1$ si $x=0$).

Exercice 2 :

1) $(k+1)^4 - k^4 = k^4 + 4k^3 + 6k^2 + 4k + 1 - k^4 = 4k^3 + 6k^2 + 4k + 1$

2) $\displaystyle S = \sum_{k=0}^{n} (k+1)^4 - k^4 = 4\sum_{k=0}^{n} k^3 + 6\sum_{k=0}^{n} k^2 + 4\sum_{k=0}^{n} k + \sum_{k=0}^{n} 1$
$\displaystyle= 4\sum_{k=0}^{n} k^3 + n(n+1)(2n+1) + 2n(n+1) + n+1$

3) Par télescopage, on a aussi $S = (n+1)^4 - 0^4 = (n+1)^4$ d'où
$\displaystyle\sum_{k=0}^{n} k^3 = \frac{1}{4}\left(S - n(n+1)(2n+1) - 2n(n+1) - n - 1\right) = \frac{1}{4}\left[(n+1)^4 - n(n+1)(2n+1) - 2n(n+1) - (n+1)\right]$
$\displaystyle= \frac{n+1}{4}\left[(n+1)^3 - n(2n+1) - 2n - 1\right] = \frac{n+1}{4}\left[n^3 + 3n^2 + 3n - 2n^2 - 3n\right] = \frac{n+1}{4}(n^3+n^2)$
$= n^2(n+1)^2/4$

Exercice 3

Via $j = k-p$ on a, puisque $k=j+p$, et que $p \leq k \leq n \Leftrightarrow 0 \leq j \leq n-p$

$$\sum_{k=p}^{n} q^k = \sum_{j=0}^{n-p} q^{j+p} = \sum_{j=0}^{n-p} q^j q^p = q^p \sum_{j=0}^{n-p} q^j = q^p \times \frac{1-q^{n-p+1}}{1-q}$$

$$= \boxed{\frac{q^p - q^{n+1}}{1-q}}$$

Exercice 4

1) $$\sum_{k=0}^{n} \binom{n}{k} 2^k 3^{n-k} = (2+3)^n = 5^n$$

2) $$\sum_{k=0}^{n} \binom{n+1}{k} 2^k 3^{n+1-k} = \sum_{k=0}^{n+1} \binom{n+1}{k} 2^k 3^{n+1-k} - \binom{n+1}{n+1} 2^{n+1} 3^0$$
$$= 5^{n+1} - 2^{n+1}$$

3) $$\sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^{k+1}}{3^{k+1}} = -\frac{1}{3} \sum_{k=0}^{n} \binom{n}{k} \left(-\frac{1}{3}\right)^k 1^{n-k} = -\frac{1}{3}\left(-\frac{1}{3}+1\right)^n$$
$$= -\frac{1}{3} \times \left(\frac{2}{3}\right)^n$$

4) $$\sum_{k=1}^{n} \binom{n}{k-1} = \sum_{j=0}^{n-1} \binom{n}{j} = \sum_{j=0}^{n} \binom{n}{j} - \binom{n}{n} = 2^n - 1$$

5) $$\sum_{k=1}^{n} \binom{n-1}{k-1} = \sum_{j=0}^{n-1} \binom{n-1}{j} = 2^{n-1}$$

6) $$\sum_{k=1}^{n} \binom{n-1}{k-1} x^k y^{n-k} = \sum_{j=0}^{n-1} \binom{n-1}{j} x^{j+1} y^{n-(j+1)}$$

via $j = k-1$ i.e. $k = j+1$

$$= x \sum_{j=0}^{n-1} \binom{n-1}{j} x^j y^{n-1-j} =$$

$$= x(x+y)^{n-1}.$$