Exercice 2 :

1) a) $\int_0^{\frac{\pi}{4}} t\cos(2t)\,dt$

$u(t) = t$ $\qquad$ $v(t) = \frac{\sin(2t)}{2}$

$u'(t) = 1$ $\qquad$ $v'(t) = \cos(2t)$

Ipp (u et v C^1 sur $\mathbb{R}$)

$$\int_0^{\frac{\pi}{4}} t\cos(2t) = \left[t \times \frac{\sin(2t)}{2}\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \frac{\sin(2t)}{2}\,dt$$

$$= \frac{\pi}{4}\,\frac{\sin\left(\frac{\pi}{2}\right)}{2} - \left[\frac{-\cos(2t)}{4}\right]_0^{\frac{\pi}{4}}$$

$$= \frac{\pi}{8} - \left(\frac{-\cos\left(\frac{2\pi}{4}\right)}{4} + \frac{\cos(0)}{4}\right)$$

$$= \frac{\pi}{8} - \frac{1}{4} = \boxed{\frac{\pi - 2}{8}}$$ ✓

b) $\int_0^1 t^2 e^{-t}\,dt$

$u(t) = t^2$ $\qquad$ $v(t) = -e^{-t}$

$u'(t) = 2t$ $\qquad$ $v'(t) = e^{-t}$

$$\int_0^1 t^2 e^{-t}\,dt = \left[t^2 \times (-e^{-t})\right]_0^1 - \int_0^1 2t(-e^{-t})\,dt$$

$$= -e^{-1} - \left(\left[2t \times (-e^{-t})\right]_0^1 - \int_0^1 2e^{-t}\,dt\right) = \boxed{2 - \frac{5}{e}}$$

$$\cancel{= -e^{-1} + 2e^{-1} + \left[-2e^{-t}\right]_0^1}$$

$$\cancel{= e^{-1} - (-2e^{-1} + 2)}$$

$$\cancel{= \boxed{3e^{-1} - 2}}$$

c) $\int_1^2 \sqrt{t}\,\ln(t)\,dt$.

$u(t) = \ln(t)$. $\qquad v(t) = \frac{2}{3} t^{\frac{3}{2}}$.

$u'(t) = \frac{1}{t}$. $\qquad v'(t) = \sqrt{t}$.

C^1 donc.

$$\int_1^2 \sqrt{t}\,\ln(t)\,dt = \left[\ln(t)\,\frac{2}{3} t^{\frac{3}{2}}\right]_1^2 - \int_1^2 \frac{1}{t} \times \frac{2}{3} t^{\frac{3}{2}}\,dt$$

$$= \ln(2)\,\frac{2}{3} \times 2\sqrt{2} - \frac{2}{3} \times \ln(1) - \frac{2}{3} \int_1^2 \frac{1}{t} \times t^{\frac{3}{2}}\,dt$$

$$= \frac{4\sqrt{2}}{3} \ln(2) - \frac{2}{3} \int_1^2 t^{\frac{1}{2}}\,dt$$

$$= \frac{4\sqrt{2}}{3} \ln(2) - \frac{2}{3} \left[\frac{2}{3} t^{\frac{3}{2}}\right]_1^2$$

$$= \frac{4\sqrt{2}}{3} \ln(2) - \frac{4}{9} \left(2^{\frac{3}{2}} - 1\right)$$

$$= \frac{4\sqrt{2}}{3} \ln(2) - \frac{4}{9} \left(2\sqrt{2} - 1\right) \quad ✓$$

2) a) $\int_0^x e^t \sin(t)\,dt$

$u(t) = e^t$ $\quad v(t) = -\cos(t)$

$u'(t) = e^t$ $\quad v'(t) = \sin(t)$

u et v C^1 sur $\mathbb{R}$.

Alors $\int_0^x e^t \sin(t)\,dt = \left[e^t \times (-\cos(t))\right]_0^x - \int_0^x e^t(-\cos(t))\,dt$

$= e^x \times (-\cos(x)) + 1 - \left[e^t \times (-\sin(t))\right]_0^x - \int_0^x e^t(-\sin(t))\,dt$

$2\int_0^x e^t \sin(t)\,dt = \dfrac{e^x \times (-\cos(x)) + 1 - e^x \times (-\sin(x)) \cancel{+ \sin(x)}}{2}$

$\cancel{e}\, e^x$

b) $\int_1^x \sin(\ln(t))\,dt$

$u(t) = t$ $\quad v(t) = \sin(\ln(t))$

$u'(t) = 1$ $\quad v'(t) = \frac{1}{t}\cos(\ln(t))$

$\int_1^x \sin(\ln(t))\,dt = \left[t \times \sin(\ln(t))\right]_1^x - \int_1^x \cancel{t}\,\cancel{\frac{1}{t}} \cos(\ln(t))\,dt$

$= x\sin(\ln(x)) - \int_1^x \cos(\ln(t))\,dt$

$u_1(t) = t$ $\quad v_1(t) = \cos(\ln(t))$

$u_1'(t) = 1$ $\quad v_1'(t) = -\frac{1}{t}\sin(\ln(t))$

$= \left[t\cos(\ln(t))\right]_1^x - \int_1^x -\sin(\ln(t))\,dt$

$2\int_1^x \sin(\ln(t))\,dt = x\sin(\ln(x)) - (x\cos\ln(x) - 1)$

$= x\sin(\ln(x)) - x\cos(\ln(x)) + 1$

✓

2

2

c) $\int_0^x e^{-t} \ln(e^{2t}+1)\, dt$.

$u(t) = -e^{-t}$ $\qquad$ $v(t) = \ln(e^{2t}+1)$

$u'(t) = e^{-t}$ $\qquad$ $v'(t) = \dfrac{2e^{2t}}{e^{2t}+1}$

$= \left[-e^{-t} \times \ln(e^{2t}+1)\right]_0^x - \int_0^x -e^{-t} \dfrac{2e^{2t}}{e^{2t}+1}\, dt$

$= -e^{-x} \times \ln(e^{2x}+1) - (-1\ln(2)) + \left[2\arctan(e^{t})\right]_0^x$

$= -e^{x} \ln(e^{2x}+1) + 2\arctan(e^{x}) - \dfrac{\pi}{2} + \ln(2)$

3) a) $\int_0^x \arctan(t)\,dt$

$= \left[t\arctan(t)\right]_0^x - \int_0^x \frac{t}{t^2+1}\,dt$

$= x\arctan(x) - \frac{1}{2}\left[\ln(t^2+1)\right]_0^x$

$= x\arctan(x) - \frac{1}{2}\ln(t^2+1)$

Les primitives de arctan sur $\mathbb{R}$ sont

$x \longmapsto x\arctan(x) - \frac{1}{2}\ln(t^2+1) + c$, $c \in \mathbb{R}$ ✓

b) $\int_1^x t^2\ln(t)\,dt$

$u(t) = \frac{t^3}{3}$ $\quad v(t) = \ln(t)$

$u'(t) = t^2$ $\quad v'(t) = \frac{1}{t}$

$\int_1^x t^2\ln(t)\,dt = \frac{1}{3}\left[\ln(t)\,t^3\right]_1^x - \frac{1}{3}\int_1^x \frac{t^{\cancel{3}}}{\cancel{3}} \times \frac{1}{\cancel{t}}\,dt$

$= \frac{1}{3}\left(\ln(x)\,x^3\right) - \frac{1}{3}\left[\frac{t^3}{3}\right]_1^x$

$= \frac{1}{3}\ln(x)\,x^3 - \frac{1}{3}\frac{x^3}{3} - \frac{1}{3}$

$= \frac{1}{3}\left(\ln(x)\,x^3 - \frac{x^3}{3} - 1\right)$

Les primitives de $x^2\ln(x)$ sur $\mathbb{R}_*^+$ sont

$x \longmapsto \frac{1}{3}\left(\ln(x)\,x^3 - \frac{x^3}{3}\ \cancel{-1}\right) + c$, $c \in \mathbb{R}$

3.

c) $\displaystyle\int_1^x \ln(t)^2 \, dt$.

$u(t) = t$ $\qquad v(t) = \ln(t)^2$

$u'(t) = 1$ $\qquad v'(t) = 2 \times \frac{1}{t} \ln(t)$

$\qquad\qquad\qquad\; = \frac{2\ln(t)}{t}$

C^1 donc.

$$\int_1^x \ln(t)^2 \, dt = \left[t \ln(t)^2\right]_1^x - \int_1^x \cancel{t}\, 2\ln(t) \, dt$$

$$= x \ln(x)^2 - 2\left[t \ln(t) - t\right]_1^x$$

$$= x \ln(x)^2 - 2\left(x \ln(x) - x + 1\right)$$

$$= x \ln(x)^2 - 2x \ln(x) + 2x - 2$$

Les primitives de $\ln(x)^2$ sur $\mathbb{R}_+^*$ sont :

$x \longmapsto x \ln(x)^2 - 2x \ln(x) + 2x \cancel{+2} + c$, $c \in \mathbb{R}$.

Exercice 3)

1) a) $\displaystyle\int_{\ln(\pi/4)}^{\ln(\pi)} e^t \cos(e^t)\, dt$ via $u = e^t$.

On a $\dfrac{du}{dt} = e^t = u$ $\Leftrightarrow$ $u \times dt = du$

Donc $e^t \cos(e^t)\, dt = \cos(u)\, du$

t	$u = e^t$
$\ln(\pi)$	π
$\ln(\pi/4)$	$\pi/4$

Donc $\displaystyle\int_{\ln(\pi/4)}^{\ln(\pi)} e^t \cos(e^t)\, dt = \int_{\pi/4}^{\pi} \cos(u)\, du$

$= \left[\sin(u)\right]_{\pi/4}^{\pi} = \sin(\pi) - \sin(\pi/4) = -\dfrac{\sqrt{2}}{2}$ ✓

b) $\displaystyle\int_1^4 \frac{(\sqrt{t}-1)^4}{\sqrt{t}}\, dt$ via $u = \sqrt{t}$.

$\dfrac{du}{dt} = \dfrac{1}{2\sqrt{t}} = \dfrac{1}{2u}$ $\Leftrightarrow$ $du = \dfrac{dt}{2u}$

Donc $\dfrac{(\sqrt{t}-1)^4}{\sqrt{t}}\, dt = 2(u-1)^4\, du$

t	$\sqrt{t} = u$
1	1
4	$\sqrt{4}$ 2

Alors $\int_1^4 \frac{(\sqrt{t}-1)^4}{\sqrt{t}}\,dt = \int_1^2 2(u-1)^4\,du$.

$= 2\left[\frac{1}{5}(u-1)^5\right]_1^2 = \frac{2}{5}\cdot\left((2-1)^5 - (1-1)^5\right)$

$= \frac{2}{5}$. ✓

c) $\int_0^{\pi/4} \sqrt{1-\sin(t)}\,dt$ via $u = \sin(t)$

$\frac{du}{dt} = \cos(t) \Leftrightarrow du = \cos(t)\,dt$.

$\cos^2(t) + \sin^2(t) = 1 \;\cancel{\Rightarrow}\; \cos(t) = \sqrt{1-\sin^2(t)}$
$= \sqrt{1-u^2}$.

(+) argu^t supplémentaire

Correc^o faite en classe.

2) a) $\displaystyle\int_1^2 \frac{\ln(x+1)-\ln(x)}{x^2}\,dx$ via $t=\frac{1}{x}$

donc

$\frac{dt}{dx} = -\frac{1}{x^2}$ $dt = -\frac{1}{x^2}\,dx.$

Donc $\displaystyle\frac{\ln(x+1)-\ln(x)}{x^2}\,dx = \ln\left(1+\frac{1}{x}\right)\times(-dt) = -\ln(1+t)\,dt$

Donc $\displaystyle\int_1^2 \frac{\ln(x+1)-\ln(x)}{x^2}\,dx = \int_1^{\frac{1}{2}} -\ln(1+t)\,dt = \int_{\frac{1}{2}}^1 \ln(1+t)\,dt$

$\displaystyle= \Big[(1+t)\ln(1+t)-(1+t)\Big]_{\frac{1}{2}}^1 = \frac{7\ln(2)-3\ln(3)-1}{2}$

b) $\displaystyle\int_0^1 \frac{1}{2+e^{-t}}\,dt$ avec $u=e^t$.

donc

$\frac{du}{dt} = e^t.$ $du = e^t\,dt$ donc $dt = \frac{du}{e^t} = \frac{du}{u}$

Donc $\displaystyle\frac{1}{2+e^{-t}}\,dt = \frac{1}{2+\frac{1}{u}}\times\frac{du}{u} = \frac{1}{1+2u}\,du$

Et

t	$u=e^t$
0	1
1	e^1

$$\text{D'où} \quad \int_0^1 \frac{1}{2+e^{-t}}\,dt = \cancel{\int_{e^{-1}}^{1} \frac{1}{1+\frac{1}{u}}\,du} \cancel{= \int_{e^{-1}}^{1} \frac{u}{u+1}\,du}$$

$$\cancel{= \int}$$

$$\int_0^1 \frac{1}{2+e^{t}}\,dt = \int_1^e \frac{1}{1+2u}\,du = \left[\frac{\ln(1+2u)}{2}\right]_1^e = \frac{1}{2}\ln\left(\frac{1+2e}{3}\right)$$

<u>Réponses de la suite de l'exo 3</u>

2.c) $\displaystyle\int_0^3 \frac{x}{\sqrt{1+x}}\,dx = \int_1^2 (u^2-1)\times 2\,du = \frac{8}{3}$

3.a) $\displaystyle\int_e^x \frac{\ln(\ln(t))}{t}\,dt = \int_1^{\ln(x)} \ln(u)\,du$

$$= \ln(x)\ln(\ln(x)) - \ln(x) + 1$$

3.b) via $u = \frac{2}{\sqrt{3}}t - \frac{1}{\sqrt{3}}$

$$\int_0^1 \frac{1}{1+\left(\frac{2}{\sqrt{3}}t - \frac{1}{\sqrt{3}}\right)^2}\,dt = \int_{-\frac{1}{\sqrt{3}}}^{\frac{1}{\sqrt{3}}} \frac{\sqrt{3}}{2} \times \frac{1}{1+u^2}\,du = \frac{\sqrt{3}\,\pi}{6}$$

3.c) $\displaystyle\int_1^x e^{\sqrt{t}}\,dt = \int_1^{\sqrt{x}} 2ue^u\,du \underset{\text{IPP}}{=} 2\left[ue^u\right]_1^{\sqrt{x}} - 2\int_1^{\sqrt{x}} e^u\,du$

$$= 2(\sqrt{x}-1)e^{\sqrt{x}}$$