TD 3 - Eléments de correction

exo 10

1) Faire une récurrence double : I : $u_0 = 2 > 0$, $u_1 = 1 > 0$

H : Si $u_n > 0$ et $u_{n+1} > 0$ alors $u_{n+2} = u_{n+1}^4 / u_n^3 > 0$

2) $(v_n) = (\ln(u_n))$ car alors $\forall n \in \mathbb{N}$:

$$v_{n+2} = \ln(u_{n+2}) = \ln\left(\frac{u_{n+1}^4}{u_n^3}\right) = \ln(u_{n+1}^4) - \ln(u_n^3) = 4\ln(u_{n+1}) - 3\ln(u_n) = 4v_{n+1} - 3v_n$$

3) (v_n) est une SRL2, $P(X) = X^2 - 4X + 3$, $q_1 = 3$, $q_2 = 1$

$$v_n = \lambda 3^n + \mu 1^n = \lambda 3^n + \mu \quad , \quad \begin{cases} v_0 = \ln(u_0) = \ln(2) = \lambda + \mu \\ v_1 = \ln(u_1) = \ln(1) = 0 = 3\lambda + \mu \end{cases} \iff \begin{cases} \lambda = -\frac{\ln(2)}{2} \\ \mu = \frac{3}{2}\ln(2) \end{cases}$$

Ainsi $v_n = \frac{3}{2}\ln(2) - \frac{\ln(2)}{2} 3^n = \frac{\ln(2)}{2} \times (3 - 3^n)$

puis $u_n = \exp(v_n) = \exp\left(\frac{3-3^n}{2} \times \ln(2)\right) = \exp\left(\ln\left(2^{\frac{3-3^n}{2}}\right)\right) = 2^{\frac{3-3^n}{2}} = \sqrt{2}^{3-3^n}$

exo 12

1) Le polynôme caractéristique est $P(X) = X^2 - 2\sqrt{3}X + 12$

Son discriminant vaut $\Delta = (-2\sqrt{3})^2 - 4 \times 12 = -36$

Ses racines sont $\frac{2\sqrt{3} + 6i}{2} = \sqrt{3} + 3i = z_1$ et $z_2 = \overline{z_1} = \sqrt{3} - 3i$

On calcule $|z_1| = \sqrt{\sqrt{3}^2 + 3^2} = \sqrt{12} = 2\sqrt{3}$

Donc $z_1 = 2\sqrt{3}\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = 2\sqrt{3}e^{i\pi/3}$

Ainsi, $\exists \lambda, \mu \in \mathbb{R} : \forall n \in \mathbb{N}$, $u_n = (2\sqrt{3})^n\left(\lambda\cos\left(\frac{n\pi}{3}\right) + \mu\sin\left(\frac{n\pi}{3}\right)\right)$

On obtient alors $\begin{cases} u_0 = (2\sqrt{3})^0(\lambda\cos(0) + \mu\sin(0)) \\ u_1 = 2\sqrt{3}\left(\lambda\cos\left(\frac{\pi}{3}\right) + \mu\sin\left(\frac{\pi}{3}\right)\right) \end{cases} \iff \begin{cases} \lambda = \sqrt{3} \\ 2\sqrt{3}\left(\lambda\frac{1}{2} + \mu\frac{\sqrt{3}}{2}\right) = 2 \end{cases}$

$\iff \begin{cases} \lambda = \sqrt{3} \\ 3 + 3\mu = 2 \end{cases} \iff \begin{cases} \lambda = \sqrt{3} \\ \mu = -\frac{1}{3} \end{cases}$ Finalement : $\forall n \in \mathbb{N}$, $u_n = (2\sqrt{3})^n\left(\sqrt{3}\cos\left(\frac{n\pi}{3}\right) - \frac{1}{3}\sin\left(\frac{n\pi}{3}\right)\right)$

2) Même rédaction : $P(X) = X^2 + 1$, $z_1 = i$, $(z_2 = -i)$, $|z_1| = 1$, $z_1 = 0 + 1i = e^{i\pi/2}$

$u_n = \lambda\cos\left(\frac{n\pi}{2}\right) + \mu\sin\left(\frac{n\pi}{2}\right) \quad \begin{cases} u_0 = \lambda = 3 \\ u_1 = \lambda\cos\left(\frac{\pi}{2}\right) + \mu\sin\left(\frac{\pi}{2}\right) = \mu = 2 \end{cases}$

$u_n = 3\cos\left(\frac{n\pi}{2}\right) + 2\sin\left(\frac{n\pi}{2}\right)$

3) On s'inspire de l'exo 10, et on pose $v_n = \ldots$

On doit trouver au final $u_n = \exp\left(\sqrt{2}^n\left(3\cos\left(\frac{n\pi}{4}\right) + (\ln(2) - 3)\sin\left(\frac{n\pi}{4}\right)\right)\right)$