TD5 :

exo 18 : On a :

$$P_n + I_n = \sum_{\substack{k=0 \\ k \text{ pair}}}^{n} \binom{n}{k} + \sum_{\substack{k=0 \\ k \text{ impair}}}^{n} \binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k} = 2^n$$

et

$$P_n - I_n = \sum_{\substack{k=0 \\ k \text{ pair}}}^{n} \binom{n}{k} - \sum_{\substack{k=0 \\ k \text{ impair}}}^{n} \binom{n}{k}$$

$$= \sum_{\substack{k=0 \\ k \text{ pair}}}^{n} \binom{n}{k}(-1)^k + \sum_{\substack{k=0 \\ k \text{ impair}}}^{n} \binom{n}{k}(-1)^k$$

$$= \sum_{k=0}^{n} \binom{n}{k}(-1)^k = 0$$

donc $P_n = I_n$.

Et comme $P_n + I_n = 2^n$, on a $2P_n = 2^n$

et finalement $P_n = I_n = 2^{n-1}$

exo 17 :

$$(1+\sqrt{2})^n + (1-\sqrt{2})^n = \sum_{k=0}^{n} \binom{n}{k}\left(\sqrt{2}^k + (-\sqrt{2})^k\right) \quad (*)$$

Or si k est impair alors $\sqrt{2}^k + (-\sqrt{2})^k = \sqrt{2}^k - \sqrt{2}^k = 0$

et si k est pair alors $\sqrt{2}^k + (-\sqrt{2})^k = 2\sqrt{2}^k = 2 \times 2^{\frac{k}{2}} \in \mathbb{N}$

Ainsi, la somme $(*)$ ne contient que les termes pairs et vaut

$$(1+\sqrt{2})^n + (1-\sqrt{2})^n = \sum_{\substack{k=0 \\ k \text{ pair}}}^{n} \binom{n}{k} \times 2 \times 2^{\frac{k}{2}} \in \mathbb{N}$$