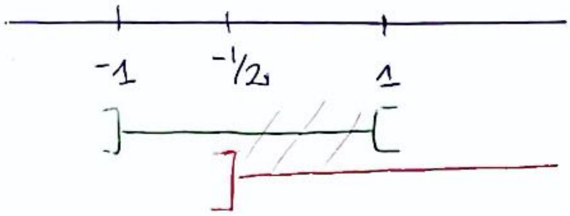
DS3 – Corrigé
Mercredi 16 Novembre 2022.

Exercice de cours

1. Ensemble de résolution:

$1-x^2>0$ et $2x+1>0$

$\iff -1<x<1$ et $x>-\frac{1}{2}$



Donc on résout sur $]-\frac{1}{2}, 1[$.

$$(1) \iff \ln(1-x^2) = \ln(2x+1)$$
$$\iff 1-x^2 = 2x+1$$
$$\iff x^2+2x=0$$
$$\iff x(x+2)=0$$
$$\iff x=0 \in]-\tfrac{1}{2},1[\text{ ou } x=-2 \notin]-\tfrac{1}{2},1[$$

Conclusion: $\boxed{S=\{0\}}$

2. $(2) \iff e^{2x}-6+e^{x}=0$ car $e^x \neq 0 \ \forall x\in\mathbb{R}$.

Posons $X=e^x>0$

donc $(2) \iff X^2+X-6=0$

2 est racine évidente donc l'autre est $-3 \notin]0,+\infty[$

Donc $(2) \iff X=2 \iff e^x=2 \iff x=\ln(2)$

Conclusion: $\boxed{S=\{\ln 2\}}$

3. On pose $Z=z^2 \in \mathbb{C}$

donc $(3) \iff 2Z^2-7Z-4=0$.

$\Delta = 49 + 2\times 4\times 4 = 81 = 9^2 > 0$

donc $(3) \Leftrightarrow Z = \frac{7-9}{4} = -\frac{1}{2}$ ou $Z = \frac{7+9}{4} = 4$

$\Leftrightarrow t^2 = -\frac{1}{2}$ ou $t^2 = 4$

$\Leftrightarrow t = i\sqrt{\frac{1}{2}}$ ou $t = -i\sqrt{\frac{1}{2}}$ ou $t = 2$ ou $t = -2$

Conclusion: $\boxed{S = \left\{\frac{i}{\sqrt{2}}, \frac{-i}{\sqrt{2}}, 2, -2\right\}.}$

4. $\forall n \in \mathbb{N}^*$,

$$\sum_{k=1}^{n-1} \binom{n-1}{k+1} 2^k \underset{k'=k+1}{=\!=\!=} \sum_{k=2}^{n} \binom{n-1}{k} 2^{k-1}$$

$$= \frac{1}{2} \sum_{k=2}^{n} \binom{n-1}{k} 2^k$$

$$= \frac{1}{2}\left(\sum_{k=2}^{n-1} \binom{n-1}{k} 2^k + \underbrace{\binom{n-1}{n}}_{=0} 2^n\right)$$

$$= \frac{1}{2}\left(\sum_{k=0}^{n-1} \binom{n-1}{k} 2^k - \binom{n-1}{0} 2^0 - \binom{n-1}{1} 2^n\right)$$

$$= \frac{1}{2}\left((1+2)^{n-1} - 1 - (n-1)2^n\right)$$

Conclusion: $\boxed{\sum_{k=1}^{n-1} \binom{n-1}{k+1} 2^k = \frac{3^{n-1} - 1 - (n-1)2^n}{2}}$

5. $z = \frac{\sqrt{3} - i}{\sqrt{3}(\sqrt{3}i - 1)} = \frac{1}{\sqrt{3}} \frac{(\sqrt{3} - i)(-\sqrt{3}i - 1)}{(\sqrt{3})^2 + (-1)^2} = \frac{-3i - \sqrt{3} + \sqrt{3} + i}{4\sqrt{3}}.$

$= \frac{-2i - 2\sqrt{3}}{4\sqrt{3}}$. Conclusion: $\boxed{z = -\frac{1}{2} - \frac{1}{2\sqrt{3}} i}$

5. (a) $\left(u_n = u_{n-1}^2 + \dfrac{(n-1)^3}{(n-1)^2+1} \right)$

```
def suite(n):
    u=5
    for k in range(2,n+1):
        u= u**2 + (k-1)**3/((k-1)**2+1)
    return u
```

(b)

```
def monotonie(n):
    s = suite(2)-suite(1)
    for k in range( 2, n    ):
        if (suite(k+1)-suite(k))*s <0:
            return False
    return True
```

Exercice 1.

1. Exercice traité en classe !

2. $\forall k \geq 2, \quad a\binom{k}{2} + b\binom{k}{1} = a\,\dfrac{k(k-1)}{2} + bk$

$$= \frac{ak^2+(2b-a)k}{2}$$

On veut $k^2 = \dfrac{ak^2+(2b-a)k}{2}$

$\Leftrightarrow \ 2k^2 = ak^2 + (2b-a)\,k$

Donc on prend : $\begin{cases} a=2 \\ 2b-a=0 \end{cases} \iff \begin{cases} a=2 \\ b=1 \end{cases}$

Conclusion : $\boxed{\forall k \geq 2, \quad k^2 = 2\binom{k}{2} + \binom{k}{1}}$

* si $k=0$: $2\binom{0}{2} + \binom{0}{1} = 2 \times 0 + 0 = 0 = 0^2$ VRAIE

* si $k=1$: $2\binom{1}{2} + \binom{1}{1} = 2 \times 0 + 1 = 1^2$ VRAIE

3. $\forall n \in \mathbb{N}$, d'après 2),

$$\sum_{k=0}^{n} k^2 = \sum_{k=0}^{n} \left[2\binom{k}{2} + \binom{k}{1}\right] = 2\sum_{k=0}^{n}\binom{k}{2} + \sum_{k=0}^{n}\binom{k}{1}$$
$$= 2\left(\sum_{k=2}^{n}\binom{k}{2} + \underbrace{\binom{0}{2} + \binom{1}{2}}_{=0}\right) + \sum_{k=1}^{n}\binom{k}{1} + \underbrace{\binom{0}{1}}_{=0}$$

Donc d'après 1. c),

$$\sum_{k=0}^{n} k^2 = 2\binom{n+1}{3} + \binom{n+1}{2} = \binom{n+1}{3} + \underbrace{\binom{n+1}{3} + \binom{n+1}{2}}_{= \binom{n+2}{3} \text{ (Pascal)}}$$

$$= \frac{(n+1)!}{3!\,(n-2)!} + \frac{(n+2)!}{3!\,(n-1)!}$$

$$= \frac{1}{6}\Big((n+1)n(n-1) + (n+2)(n+1)n\Big)$$

$$= \frac{n(n+1)}{6}\Big(n-1+n+2\Big)$$

On retrouve bien : $\boxed{\sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}}$

Exercice 2 :

1.

```
def prod(n):
    p = 1
    for k in range(2, n+1):
        p = p * (k**3+1)/(k**3-1)
    return p.
```

2.

$\Delta = -3 < 0$

donc l'équation a 2 racines complexes conjuguées :

$$\frac{-1-i\sqrt{3}}{2} = \bar{j} \quad \text{et} \quad \frac{-1+i\sqrt{3}}{2} = j$$

Conclusion : $\boxed{S = \{j \,;\, \bar{j}\}.}$

3. Formule de Bernoulli ("ou on part de la droite") :

$$z^3 - 1 = (z-1)(z^2+z+1) = (z-1)(z-j)(z-\bar{j}) \quad \text{or } \bar{j} = j^2$$

donc $z^3 - 1 = (z-1)(z-j)(z-j^2)$

$$(z+1)(z+j)(z+j^2) = (z+1)(z+j)(z+\bar{j})$$
$$= (z+1)\left(z^2 + z(j+\bar{j}) + j\bar{j}\right)$$

or $j + \bar{j} = 2\,\mathrm{Re}(j) = 2 \times \left(-\frac{1}{2}\right) = -1$ et $j\bar{j} = |j|^2 = 1$.

donc $(z+1)(z+j)(z+j^2) = (z-1)(z^2-z+1) = (z+1)(z^2-z+1)$

$$= z^3 + 1.$$

4.

$$P = \prod_{k=2}^{n} \frac{(k+1)(k+j)(k+j^2)}{(k-1)(k-j)(k-j^2)} = \left(\prod_{k=2}^{n} \frac{k+1}{k-1}\right) \prod_{k=2}^{n} \frac{(k+j)(k+j^2)}{(k-j)(k-j^2)}$$

Or $\quad k - j^2 = -j-1 \quad (cf\ 2)$

donc $\quad k-j^2 = k+j+1 = j+k+1$

et $k+j^2 = k-j-1 = -j+k-1$

Donc $\boxed{P = \left(\prod_{k=2}^{n} \frac{k+1}{k-1}\right)\left(\prod_{k=2}^{n} \frac{(j+k)}{j+k+1}\right)\left(\prod_{k=2}^{n} \frac{-j+k-1}{-j+k}\right)}$

(5.) On télescope chaque produit :

$$* \prod_{k=2}^{n} = \frac{\prod_{k=2}^{n}(k+1)}{\prod_{k=2}^{n}(k-1)} = \frac{3\times 4\times .. \times (n+1)}{1\times 2\times ..\times(n-1)} = \frac{n(n+1)}{2}.$$

$$* \prod_{k=2}^{n} \frac{j+k}{j+k+1} = \frac{\prod_{k=2}^{n}(j+k)}{\prod_{k=3}^{n+1}(j+k)} = \frac{j+2}{j+(n+1)}$$

$$* \prod_{k=2}^{n} \frac{-j+k-1}{-j+k} = \frac{\prod_{k=1}^{n-1}(-j+k)}{\prod_{k=2}^{n}(-j+k)} = \frac{-j+1}{-j+n}$$

Donc $\quad P = \frac{n(n+1)}{2} \times \frac{(j+2)(1-j)}{(n-j)(n+1+j)}$

avec $(j+2)(1-j) = 2-j^2-j = 2-(-1)$ car $j^2+j+1=0$.
$= 3$

et $(n-j)(n+1+j) = n(n+1) + jn - j(n+1) - j^2$
$= n(n+1) - j - j^2$
$= n(n+1) + 1 = n^2+n+1$

<u>Conclusion</u> : $\boxed{P = \frac{3n(n+1)}{2(n^2+n+1)}}$

Exercice 3 :

(1.) $e^{ia}+e^{ib}+e^{ic} = \cos a + i\sin a + \cos b + i\sin b + \cos c + i\sin c$

$= (\cos a + \cos b + \cos c) + i(\sin a + \sin b + \sin c)$

$= 0 + i0$

donc $e^{ia}+e^{ib}+e^{ic} = 0$

(2.) (a) $e^{ia}+e^{ib}+e^{ic} = e^{ic}\left(e^{i(a-c)} + e^{i(b-c)} + 1\right) = 0$

or $e^{ic} \neq 0$ donc $1+e^{i\alpha}+e^{i\beta} = 0$

(b) $1+e^{i\alpha}+e^{i\beta} = (1+\cos\alpha+\cos\beta) + i(\sin\alpha+\sin\beta) = 0$

donc $\sin\alpha + \sin\beta = 0 \iff \sin\alpha = -\sin\beta = \sin(-\beta)$

$\iff \alpha = -\beta + 2k\pi$, ou $\alpha = \pi - (-\beta) + 2k\pi$ avec $k\in\mathbb{Z}$.

Conclusion : $\alpha = -\beta + 2k\pi$ ou $\alpha = \pi + \beta + 2k\pi$, avec $k \in \mathbb{Z}$

(c) $(1+\cos\alpha+\cos\beta) + i(\sin\alpha+\sin\beta) = 0$

donc $1+\cos\alpha+\cos\beta = 0$

Par l'absurde : si $\exists k / \alpha = \pi+\beta+2k\pi$ alors $\cos\alpha = \cos(\beta+\pi) = -\cos\beta$.

donc $1+\cos\alpha+\cos\beta = 1+\cos\beta-\cos\beta = 1$

donc $1 = 0$: ABSURDE

Conclusion : $\forall k\in\mathbb{Z},\ \alpha \neq \pi+\beta+2k\pi$

(d) Puisque $\alpha = -\beta + 2k\pi$, $k\in\mathbb{Z}$, $\cos(\alpha) = \cos(-\beta) = \cos(\beta)$

donc $1+\cos\alpha+\cos\beta = 1 + 2\cos\beta = 0$

donc $\boxed{\cos\beta = -\frac{1}{2} = \cos\alpha}$

et $\sin^2(\alpha) = 1 - \cos^2\alpha$

donc $|\sin(\alpha)| = \sqrt{1-\cos^2\alpha}$. Or $\sin\alpha \geq 0$

donc $\boxed{\sin\alpha = \sqrt{1-\frac{1}{4}} = \frac{\sqrt{3}}{2}}$

donc $\boxed{\sin(\beta) = \sin(-\alpha) = -\sin\alpha = -\frac{\sqrt{3}}{2}}$

donc $e^{i\alpha} = \cos\alpha + i\sin\alpha = -\frac{1}{2} + i\frac{\sqrt{3}}{2} = j$

donc $e^{i(a-c)} = e^{ia}e^{-ic} = j$, soit $\boxed{e^{ia} = j\,e^{ic}}$

et $e^{i\beta} = \cos\beta + i\sin\beta = -\frac{1}{2} - i\frac{\sqrt{3}}{2} = \bar{j} = j^2$

donc $e^{i(b-c)} = j^2$, soit $\boxed{e^{ib} = j^2 e^{ic}}$

3. $e^{i2a} = \left(e^{ia}\right)^2 = \left(j\,e^{ic}\right)^2 = j^2 e^{2ic}$

et $e^{i2b} = \left(e^{ib}\right)^2 = j^4 e^{2ic} = \underbrace{j^3}_{=1} j\, e^{2ic} = j\,e^{2ic}$

donc $e^{i(2a)} + e^{i(2b)} + e^{i(2c)} = \underbrace{(j^2+j+1)}_{=0} e^{2ic} = 0.$

donc en égalant les parties réelle et imaginaire:

$$\begin{cases} \cos(2a) + \cos(2b) + \cos(2c) = 0 \\ \sin(2a) + \sin(2b) + \sin(2c) = 0. \end{cases}$$

4. * si $n = 3k+1$, $k \in \mathbb{Z}$:

$$e^{ina} + e^{inb} + e^{inc} = e^{i(3k+1)a} + e^{i(3k+1)b} + e^{i(3k+1)c}$$
$$= e^{i(3k+1)c} j^{3k+1} + e^{i(3k+1)c} j^{2(3k+1)} + e^{i(3k+1)c}$$
$$= e^{i(3k+1)c}\left(j^{3k+1} + j^{2(3k+1)} + 1\right)$$

or $j^{3k+1} + j^{2(3k+1)} + 1 = j\left(j^3\right)^k + j^2\left(j^3\right)^{2k} + 1 = j + j^2 + 1 = 0.$

donc $e^{ina} + e^{inb} + e^{inc} = 0$ donc $\begin{cases} \cos(na) + \cos(nb) + \cos(nc) = 0 \\ \sin(na) + \sin(nb) + \sin(nc) = 0. \end{cases}$

* si $n = 3k+2$, $k \in \mathbb{Z}$.

de même : $e^{ina}+e^{inb}+e^{inc}=e^{i(3k+2)c}\left(j^{3k+2}+j^{2(3k+2)}+1\right)$

or $j^{3k+2}+j^{2(3k+2)}+1=(j^3)^k j^2+(j^3)^{2k}\underbrace{j^4}_{=j^3\times j=j}+1$

$=j^2+j+1=0$

donc $\boxed{\begin{cases}\cos(na)+\cos(nb)+\cos(nc)=0\\ \sin(na)+\sin(nb)+\sin(nc)=0.\end{cases}}$

Exercice 4.

1. z est bien défini ssi :

$1+\sin(2\theta)\geqslant 0$ et $1-\sin(2\theta)\geqslant 0$

et $\sqrt{1+\sin(2\theta)}+i\sqrt{1-\sin(2\theta)}\neq 0$.

Or : $-1\leqslant\sin(2\theta)\leqslant 1$ donc $\begin{cases}1+\sin(2\theta)\geqslant 0\\ 1-\sin(2\theta)\geqslant 0\end{cases}$ ✓

et $\sqrt{1+\sin(2\theta)}+i\sqrt{1-\sin(2\theta)}=0$

$\Leftrightarrow\begin{cases}\sqrt{1+\sin(2\theta)}=0\\ \sqrt{1-\sin(2\theta)}=0\end{cases}\Leftrightarrow\begin{cases}1+\sin(2\theta)=0\\ 1-\sin(2\theta)=0\end{cases}\Leftrightarrow\begin{cases}\sin(2\theta)=-1\\ \sin(2\theta)=1\end{cases}$ IMPOSSIBLE

donc $\sqrt{1+\sin(2\theta)}+i\sqrt{1-\sin(2\theta)}\neq 0$ ✓

Conclusion : $\boxed{z \text{ est bien défini.}}$

2. $1+\cos\theta+i\sin\theta=1+e^{i\theta}=e^{i\theta/2}\left(e^{-i\theta/2}+e^{i\theta/2}\right)$

$=2\cos\left(\frac{\theta}{2}\right)e^{i\theta/2}$

or $\theta\in[0,\pi[$ donc $\frac{\theta}{2}\in\left[0,\frac{\pi}{2}\right[$ donc $\cos\left(\frac{\theta}{2}\right)>0$

donc la forme exponentielle est : $\boxed{2\cos\left(\frac{\theta}{2}\right)e^{i\frac{\theta}{2}}}$

3. (a) $(\cos\theta + \sin\theta)^2 = \cos^2\theta + \underbrace{2\cos\theta\sin\theta}_{\sin(2\theta)} + \sin^2\theta = 1 + \sin(2\theta)$ ✓

donc de même : $(\cos\theta - \sin\theta)^2 = 1 - \sin(2\theta)$

(b) $\cos\theta + \sin\theta = \sqrt{2}\left(\frac{\sqrt{2}}{2}\cos\theta + \frac{\sqrt{2}}{2}\sin\theta\right)$

$= \sqrt{2}\left(\cos\left(\frac{\pi}{4}\right)\cos\theta + \sin\left(\frac{\pi}{4}\right)\sin\theta\right)$

$= \sqrt{2}\,\cos\left(\theta - \frac{\pi}{4}\right)$

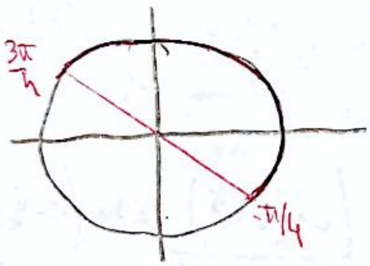
$(\sqrt{2} > 0)$ ✓

et $\cos\theta - \sin\theta = \sqrt{2}\left(\frac{\sqrt{2}}{2}\cos\theta - \frac{\sqrt{2}}{2}\sin\theta\right)$

$= \sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right)\cos\theta + \sin\left(-\frac{\pi}{4}\right)\sin\theta\right)$

$= \sqrt{2}\,\cos\left(\theta - \left(-\frac{\pi}{4}\right)\right) = \cos\left(\theta + \frac{\pi}{4}\right)$

(c) • $0 \leq x < \pi$ donc $-\frac{\pi}{4} \leq x - \frac{\pi}{4} < \frac{3\pi}{4}$.



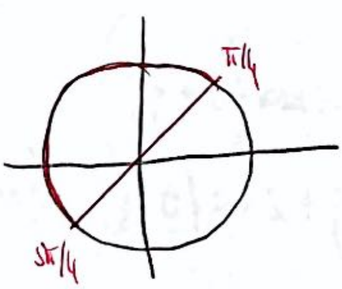
donc $\cos\left(x - \frac{\pi}{4}\right) \geq 0$

$\Leftrightarrow -\frac{\pi}{4} \leq x - \frac{\pi}{4} \leq \frac{\pi}{2}$

$\Leftrightarrow 0 \leq x \leq \frac{\pi}{2} + \frac{\pi}{4}$

Conclusion : $\boxed{S = \left[0, \frac{3\pi}{4}\right]}$

• $0 \leq x < \pi$ donc $\frac{\pi}{4} \leq x + \frac{\pi}{4} < \frac{5\pi}{4}$



$\cos\left(x + \frac{\pi}{4}\right) \geq 0 \Leftrightarrow \frac{\pi}{4} \leq x + \frac{\pi}{4} \leq \frac{\pi}{2}$

$\Leftrightarrow 0 \leq x \leq \frac{\pi}{2} - \frac{\pi}{4}$

donc $\boxed{S = \left[0, \frac{\pi}{4}\right]}$

4. $\sqrt{1+\sin(2\theta)} + i\sqrt{1-\sin(2\theta)}$

$$\underset{3.(a)}{=} \sqrt{(\cos\theta+\sin\theta)^2} + i\sqrt{(\cos\theta-\sin\theta)^2}$$

$$= |\cos\theta+\sin\theta| + i|\cos\theta-\sin\theta|$$

$$\underset{3.(b)}{=} \sqrt{2}\left|\cos\left(\theta-\frac{\pi}{4}\right)\right| + \sqrt{2}\, i\left|\cos\left(\theta+\frac{\pi}{4}\right)\right|$$

D'après 3.(c) :

θ	0		$\pi/4$		$3\pi/4$		π
$\cos(\theta-\pi/4)$		+		+	0	−	
$\cos(\theta+\frac{\pi}{4})$		+	0	−		−	

Donc trois cas :

* si $\theta \in [0, \frac{\pi}{4}]$:

$$\sqrt{1+\sin(2\theta)} + i\sqrt{1-\sin(2\theta)} = \sqrt{2}\left(\cos\left(\theta-\frac{\pi}{4}\right) + i\underbrace{\cos\left(\theta+\frac{\pi}{4}\right)}_{\sin\left(\frac{\pi}{2}-\left(\theta+\frac{\pi}{4}\right)\right) = \sin\left(-\theta+\frac{\pi}{4}\right) = -\sin\left(\theta-\frac{\pi}{4}\right)}\right)$$

$$= \sqrt{2}\left(\cos\left(\theta-\frac{\pi}{4}\right) - i\sin\left(\theta-\frac{\pi}{4}\right)\right)$$

$$= \sqrt{2}\, e^{-i(\theta-\pi/4)}$$

$$= \sqrt{2}\, e^{-i(\theta-\pi/4)}$$

* si $\theta \in \left]\frac{\pi}{4}, \frac{3\pi}{4}\right[$:

$$\sqrt{1+\sin(2\theta)} + i\sqrt{1-\sin(2\theta)} = \sqrt{2}\left(\cos\left(\theta-\frac{\pi}{4}\right) - i\cos\left(\theta+\frac{\pi}{4}\right)\right)$$

$$= \sqrt{2}\left(\cos\left(\theta+\frac{\pi}{4}\right) + i\sin\left(\theta-\frac{\pi}{4}\right)\right) \text{ (voir calcul précédent)}$$

$$= \sqrt{2}\, e^{i(\theta-\pi/4)}$$

* si $\theta \in \left[\frac{3\pi}{4}, \pi\right[$:

$$\sqrt{1+\sin(2\theta)} + i\sqrt{1-\sin(2\theta)} = \sqrt{2}\left(-\cos\left(\theta-\frac{\pi}{4}\right) + i\cos\left(\theta+\frac{\pi}{4}\right)\right)$$
$$= \sqrt{2}\left(-\cos\left(\theta-\frac{\pi}{4}\right) + i\sin\left(\theta-\frac{\pi}{4}\right)\right)$$
$$= -\sqrt{2}\, e^{-i(\theta-\pi/4)}$$
$$= \sqrt{2}\, e^{i\pi} e^{-i(\theta-\pi/4)}$$
$$= \sqrt{2}\, e^{i\left(\frac{5\pi}{4}-\theta\right)}$$

5. * si $\theta \in \left[0, \frac{\pi}{4}\right]$: $$z = \frac{2\cos\left(\frac{\theta}{2}\right) e^{i\frac{\theta}{2}}}{\sqrt{2}\, e^{-i(\theta-\pi/4)}}$$

donc $$\boxed{z = \sqrt{2}\cos\left(\frac{\theta}{2}\right) e^{i\left(\frac{3\theta}{2}-\frac{\pi}{4}\right)}}$$

* si $\theta \in \left]\frac{\pi}{4}, \frac{3\pi}{4}\right[$: $$z = \frac{2\cos\left(\frac{\theta}{2}\right) e^{i\frac{\theta}{2}}}{\sqrt{2}\, e^{i(\theta-\pi/4)}}$$ donc $$\boxed{z = \sqrt{2}\cos\left(\frac{\theta}{2}\right) e^{i\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}}$$

* si $\theta \in \left[\frac{3\pi}{4}, \pi\right[$: $$z = \frac{2\cos\left(\frac{\theta}{2}\right) e^{i\frac{\theta}{2}}}{\sqrt{2}\, e^{i\left(\frac{5\pi}{4}-\theta\right)}}$$ donc $$\boxed{z = \sqrt{2}\cos\left(\frac{\theta}{2}\right) e^{i\left(\frac{3\theta}{2}-\frac{5\pi}{4}\right)}}$$