DM3 - Corrigé

Exercice 1.

$\forall n \in \mathbb{N}$, $(1+i\sqrt{3})^n + (1-i\sqrt{3})^n = \left(2e^{i\pi/3}\right)^n + \left(2e^{-i\pi/3}\right)^n$

$= 2^n e^{i\frac{n\pi}{3}} + 2^n e^{-i\frac{n\pi}{3}}$

$= 2^n \left(e^{i\left(\frac{n\pi}{3}\right)} + e^{-i\left(\frac{n\pi}{3}\right)}\right)$

$= 2^n \times 2\cos\left(\frac{n\pi}{3}\right)$.

Conclusion : $\boxed{\forall n \in \mathbb{N},\ (1+i\sqrt{3})^n + (1-i\sqrt{3})^n = 2^{n+1}\cos\left(\frac{n\pi}{3}\right)}$

Exercice 2 : $\forall n \in \mathbb{N}$,

$$\sum_{0\leq i,j\leq n}\binom{j}{i} = \sum_{0\leq i\leq j\leq n}\binom{j}{i} + \underbrace{\sum_{0\leq j<i\leq n}\binom{j}{i}}_{=0 \text{ car } \forall i>j,\ \binom{j}{i}=0.}$$

$$= \sum_{j=0}^{n}\left(\sum_{i=0}^{j}\binom{j}{i}\right)$$

$$= \sum_{j=0}^{n} 2^j \underset{(2\neq 1)}{=} \frac{1-2^{n+1}}{1-2}$$

Conclusion : $\boxed{\forall n \in \mathbb{N},\ \sum_{0\leq i,j\leq n}\binom{j}{i} = 2^{n+1}-1}$

Exercice 3 : $\forall n \in \mathbb{N}^*$, $S_n = \sum_{k=1}^{n}\sin(n-k) = \sum_{k=1}^{n}\mathrm{Im}\left(e^{i(n-k)}\right) = \mathrm{Im}\left(\sum_{k=1}^{n}e^{i(n-k)}\right)$

Avec $\sum_{k=1}^{n}e^{i(n-k)} = e^{in}\sum_{k=1}^{n}\left(e^{-i}\right)^k \underset{e^{-i}\neq 1}{=} e^{in}\frac{e^{-i}-e^{-i(n+1)}}{1-e^{-i}}$

$$= e^{i(n-1)}\frac{1-e^{-in}}{1-e^{-i}}$$

Avec : $$1 - e^{-in} = e^{-i\frac{n}{2}}\left(e^{i\frac{n}{2}} - e^{-i\frac{n}{2}}\right)$$

$$= 2i\sin\left(\frac{n}{2}\right)e^{-i\frac{n}{2}}$$

et $$1 - e^{-i} = e^{-\frac{i}{2}}\left(e^{\frac{i}{2}} - e^{\frac{-i}{2}}\right) = 2i\sin\left(\frac{1}{2}\right)e^{-i/2}$$

Donc $$\sum_{k=1}^{n} e^{i(n-k)} = e^{i(n-1)}\,\frac{\cancel{2i}\sin\left(\frac{n}{2}\right)e^{-i\frac{n}{2}}}{\cancel{2i}\sin\left(\frac{1}{2}\right)e^{-i/2}}$$

$$= \frac{\sin\left(\frac{n}{2}\right)}{\sin\left(\frac{1}{2}\right)}\,e^{i\left(n-1-\frac{n}{2}+\frac{1}{2}\right)} = \frac{\sin\left(\frac{n}{2}\right)}{\sin\left(\frac{1}{2}\right)}\,e^{i\frac{n-1}{2}}$$

Conclusion : $$\boxed{\forall n\in\mathbb{N}^*,\ S_n = \frac{\sin\left(\frac{n}{2}\right)\sin\left(\frac{n-1}{2}\right)}{\sin\left(\frac{1}{2}\right)}}$$