DM5_ Corrigé

Exercice 1

$$(S) \underset{L_1 \leftrightarrow L_3}{\Longleftrightarrow} \begin{cases} 2x - 3y - \lambda z = 0 \\ -2x + (5-\lambda)y - 2z = 0 \\ -\lambda x + 3y + 2z = 0 \end{cases}$$

$$\underset{\substack{L_2 \leftarrow L_2 + L_1 \\ L_3 \leftarrow 2L_3 + \lambda L_1}}{\Longleftrightarrow} \begin{cases} 2x - 3y - \lambda z = 0 \\ (2-\lambda)y + (-2-\lambda)z = 0 \\ 3(2-\lambda)y + (2-\lambda)(2+\lambda)z = 0 \end{cases}$$

$$\underset{L_3 \leftarrow L_3 - 3L_2}{\Longleftrightarrow} \begin{cases} 2x - 3y - \lambda z = 0 \\ (2-\lambda)y - (2+\lambda)z = 0 \\ P(\lambda)z = 0 \end{cases}$$

avec
$$\begin{aligned} P(\lambda) &= (2-\lambda)(2+\lambda) + 3(2+\lambda) \\ &= (2+\lambda)(2-\lambda+3) \\ &= (2+\lambda)(-5-\lambda) \\ &= -(2+\lambda)(5-\lambda) \end{aligned}$$

* si $\lambda = -2$

$$(S) \Longleftrightarrow \begin{cases} 2x - 3y - 2z = 0 \\ -4z = 0 \\ 0 = 0 \end{cases} \Longleftrightarrow \begin{cases} 2x - 3y - 2z = 0 \\ -4z = 0 \end{cases}$$

donc $\boxed{rg(S) = 2}$

* si $\lambda = 5$

$$(S) \Longleftrightarrow \begin{cases} 2x - 3y - 5z = 0 \\ -3y - 7z = 0 \\ 0 = 0 \end{cases}$$

$\boxed{rg(S) = 2}$

* si $\alpha \in \mathbb{R} \setminus \{5, -2\}$, $rg(y) = 3$

Exercice 2

Partie I

1. $P^2 = \frac{1}{3}(3P) = P$ donc (par récurrence) : $\forall n \in \mathbb{N}^*, P^n = P$

$Q^2 = (I-P)^2 \underset{IP=PI}{=} I^2 - 2P + P^2 = I - 2P + P = I - P = Q$

donc (par récurrence) : $\forall n \in \mathbb{N}^*, Q^n = Q$

2. $$P + \alpha Q = P + \alpha I - \alpha P$$
$$= (1-\alpha) P + \alpha I = \begin{pmatrix} \alpha + \frac{1-\alpha}{3} & \frac{1-\alpha}{3} & \frac{1-\alpha}{3} \\ \frac{1-\alpha}{3} & \alpha + \frac{1-\alpha}{3} & \frac{1-\alpha}{3} \\ \frac{1-\alpha}{3} & \frac{1-\alpha}{3} & \alpha + \frac{1-\alpha}{3} \end{pmatrix}$$

donc on prend α tq $\begin{cases} \frac{1-\alpha}{3} = \frac{1}{6} \\ \alpha + \frac{1-\alpha}{3} = \frac{2}{3} \end{cases}$ soit $\alpha = \frac{2}{3} - \frac{1}{6} = \frac{3}{6} = \frac{1}{2}$

$$\boxed{\alpha = \frac{1}{2}}$$

3. $PQ = P(I-P) = P - P^2 = P - P = 0$

$QP = (I-P)P = P - P^2 = P - P = Q$

donc P et Q commutent et par le binôme de Newton :

$\forall n \in \mathbb{N},\quad M^n = \sum_{k=0}^{n} \binom{n}{k} \alpha^k Q^k P^{n-k}$

$\underset{\text{produit nul}}{=} \binom{n}{0} \alpha^0 Q^0 P^n + \binom{n}{n} \alpha^n Q^n P^0$

$= P^n + \alpha^n Q^n$

donc $\boxed{\forall n \in \mathbb{N}^*,\ M^n = P + \alpha^n Q}$

4.

$$\forall n \in \mathbb{N}^*,\ M^n = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + \frac{\alpha^n}{3}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

$$\boxed{M^n = \frac{1}{3}\begin{pmatrix} 1+2\alpha^n & 1-\alpha^n & 1-\alpha^n \\ 1-\alpha^n & 1+2\alpha^n & 1-\alpha^n \\ 1-\alpha^n & 1-\alpha^n & 1+2\alpha^n \end{pmatrix}}$$

Partie II :

1. $V_{n+1} = \begin{pmatrix} \frac{2}{3} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{2}{3} \end{pmatrix} V_n$ donc $\boxed{V_{n+1} = M V_n,\ \forall n \in \mathbb{N}^*}$

2. Par récurrence immédiate

3. $\forall n \in \mathbb{N}^*,\ M^{n-1} = \left(\frac{1}{3}\right)\begin{pmatrix} 1+2\alpha^{n-1} & 1-\alpha^{n-1} & 1-\alpha^{n-1} \\ 1-\alpha^{n-1} & 1+2\alpha^{n-1} & 1-\alpha^{n-1} \\ 1-\alpha^{n-1} & 1-\alpha^{n-1} & 1+2\alpha^{n-1} \end{pmatrix}$

$$\underset{(\alpha=\frac{1}{2})}{=} \left(\frac{1}{3}\right)\begin{pmatrix} 1+\left(\frac{1}{2}\right)^{n-2} & 1-\left(\frac{1}{2}\right)^{n-1} & 1-\left(\frac{1}{2}\right)^{n-1} \\ 1-\left(\frac{1}{2}\right)^{n-1} & 1+\left(\frac{1}{2}\right)^{n-2} & 1-\left(\frac{1}{2}\right)^{n-1} \\ 1-\left(\frac{1}{2}\right)^{n-1} & 1-\left(\frac{1}{2}\right)^{n-1} & 1+\left(\frac{1}{2}\right)^{n-2} \end{pmatrix}$$

$\forall n \in \mathbb{N}^*$, $$V_n = \frac{1}{3}\begin{pmatrix} \left(1-\left(\frac{1}{2}\right)^{n-1}\right)q_1 + \left(1-\left(\frac{1}{2}\right)^{n-1}\right)r_1 \\ \left(1+\left(\frac{1}{2}\right)^{n-2}\right)q_1 + \left(1-\left(\frac{1}{2}\right)^{n-1}\right)r_1 \\ \left(1+\left(\frac{1}{2}\right)^{n-2}\right)r_1 + \left(1-\left(\frac{1}{2}\right)^{n-1}\right)q_1 \end{pmatrix}$$

Or $\frac{1}{2} \in]-1,1[$ donc $\lim\limits_{n \to +\infty} \left(\frac{1}{2}\right)^n = 0$

et donc : $$p_n = \frac{1}{3}\left[-\left(\frac{1}{2}\right)^{n-1} q_1 - \left(\frac{1}{2}\right)^{n-1} r_1\right] \xrightarrow[n \to +\infty]{} \frac{q_1+r_1}{3}1 = \frac{1}{3}.$$

$$q_n = \frac{1}{3}\left[\left(1+\left(\frac{1}{2}\right)^{n-2}\right) q_1 + \left(1-\left(\frac{1}{2}\right)^{n-1}\right) r_1\right] \xrightarrow[n \to +\infty]{} \frac{1}{3}(q_1+r_1) = \frac{2}{3}$$

$$r_n = \frac{1}{3}\left[\left(1+\left(\frac{1}{2}\right)^{n-2}\right) r_1 + \left(1-\left(\frac{1}{2}\right)^{n-1}\right) q_1\right] \xrightarrow[n \to +\infty]{} \frac{1}{3}(r_1+q_1) = \frac{2}{3}.$$