DS3_ Corrigé MATHS
Mercredi 15 Novembre 2023.

Exercice 2

- $f'(x) = 4\left(\ln(\sqrt{x})\right)^3 \times \dfrac{\frac{1}{2\sqrt{x}}}{\sqrt{x}} = \cancel{4}^{\,2}\left(\ln(\sqrt{x})\right)^3 \times \dfrac{1}{\cancel{2}x}$

$$\boxed{f'(x) = \frac{2}{x} \times \left(\ln(\sqrt{x})\right)^2 \times \ln(\sqrt{x}).}$$

- si $x > 0$: $g(x) = \dfrac{x^2 e^{-x}}{\sqrt{x^2+1}}$

$$g'(x) = \frac{\left(2xe^{-x} + x^2(-1)e^{-x}\right)\sqrt{x^2+1} - x^2 e^{-x} \times \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+1}}}{\left(\sqrt{x^2+1}\right)^2}$$

$$= \frac{1}{\left(\sqrt{x^2+1}\right)^2}\left[e^{-x}\sqrt{x^2+1}\left(2x - x^2\right) - \frac{x^3 e^{-x}}{\sqrt{x^2+1}}\right]$$

$$= \frac{xe^{-x}}{\left(\sqrt{x^2+1}\right)^2}\left((2-x)\sqrt{x^2+1} - \frac{x^2}{\sqrt{x^2+1}}\right)$$

$$= \frac{xe^{-x}}{\left(\sqrt{x^2+1}\right)^3}\left((2-x)(x^2+1) - x^2\right)$$

$$\boxed{g'(x) = \frac{xe^{-x}}{\left(\sqrt{x^2+1}\right)^3}.\left(x^2 + 2 - x^3 - x\right)}$$

- si $x < 0$: $g(x) = \dfrac{x^2 e^{x}}{\sqrt{x^2+1}}$

$$g'(x) = \frac{\left(2xe^x + x^2e^x\right)\sqrt{x^2+1} - x^2e^x\,\frac{2x}{2\sqrt{x^2+1}}}{\left(\sqrt{x^2+1}\right)^2}$$

$$g'(x) = \frac{1}{\left(\sqrt{x^2+1}\right)^2}\left[xe^x(2+x)\sqrt{x^2+1} - \frac{x^3e^x}{\sqrt{x^2+1}}\right]$$

$$= \frac{xe^x}{\left(\sqrt{x^2+1}\right)^3}\left((x^2+1)(2+x) - x^2\right)$$

donc $\boxed{g'(x) = \frac{xe^x}{\left(\sqrt{x^2+1}\right)^3}\left(x^2+x^3+2+x\right)}$

Exercice 3

(1) Ensemble de résolution : $x+5 \neq 0$ et $\frac{x^2-1}{x+5} > 0$

x	$-\infty$		-5		-1		1		$+\infty$
x^2-1		$+$	|	$+$	0	$-$	0	$+$	
$x+5$		$-$	0	$+$	|	$+$	|	$+$	
$\frac{x^2-1}{x+5}$		$-$	||	$+$	0	$-$	0	$+$	

Donc on résout sur $]-5,-1[\,\cup\,]1,+\infty[$:

$$(1) \Leftrightarrow \frac{x^2-1}{x+5} \geqslant 1 \Leftrightarrow \frac{x^2-1}{x+5} - 1 \geqslant 0 \Leftrightarrow \frac{x^2-1-x-5}{x+5} \geqslant 0$$

$$\Leftrightarrow \frac{x^2-x-6}{x+5} \geqslant 0.$$

x	$-\infty$		-5		-2		3		$+\infty$
x^2-x-6		$+$		$+$	0	$-$	0	$+$	
$x+5$		$-$	0	$+$		$+$		$+$	
$\dfrac{x^2-x-6}{x+5}$		$-$	$\Vert$	$+$	0	$-$	0	$+$	

Conclusion : $\boxed{\mathcal{S} =]-5,-2] \cup [3,+\infty[}$

(2) $(2) \iff \sin^2(x) - \sin\left(2\times\frac{x}{2}\right) - 1 \geqslant 0$

$\iff \sin^2(x) - \sin(x) - 1 \geqslant 0.$

Posons $X = \sin(x) \in [-1,1]$. Donc $(2) \iff X^2 - X - 1 \geqslant 0.$

$\Delta = 1+4 = 5 > 0.$

X	$-\infty$	-1		$\frac{1-\sqrt{5}}{2}$		1	$\frac{1+\sqrt{5}}{2}$		$+\infty$
X^2-X-1			$+$	0	$-$		0	$+$	

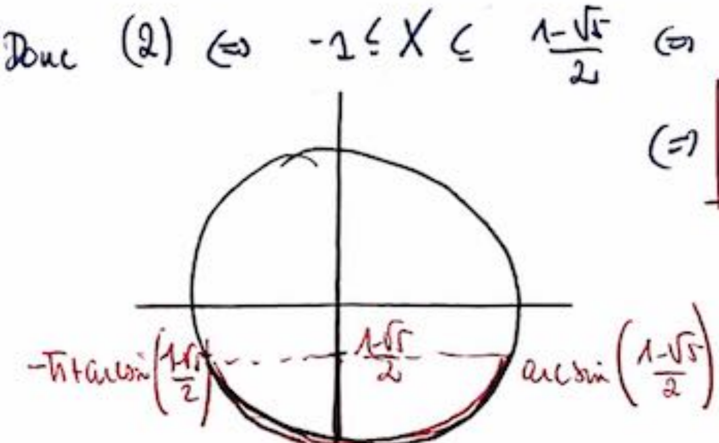
Or :

$\left(\frac{1+\sqrt{5}}{2} > 0\right)$ $\quad * \; \frac{1+\sqrt{5}}{2} \leq 1 \iff 1+\sqrt{5} \leq 2 \iff \sqrt{5} \leq 1 \iff 5 \leq 1$ FAUX

$\left(\frac{1-\sqrt{5}}{2} < 0\right)$ $\quad * \; \frac{1-\sqrt{5}}{2} \geqslant -1 \iff 1-\sqrt{5} \geqslant -2 \iff -\sqrt{5} \geqslant -3 \iff \sqrt{5} \leq 3$
$\iff 5 \leq 9$ VRAI

Donc $(2) \iff -1 \leq X \leq \frac{1-\sqrt{5}}{2} \iff -1 \leq \sin(x) \leq \frac{1-\sqrt{5}}{2}.$

$\iff \boxed{-\pi + \arcsin\left(\frac{1-\sqrt{5}}{2}\right) \leq x \leq \arcsin\left(\frac{1-\sqrt{5}}{2}\right)}$



Exercice 4

① (a)

$\forall n \in \mathbb{N}, \quad a_{n+2} = \dfrac{3a_{n+1} + b_{n+1}}{4} = \dfrac{3a_{n+1} + \frac{a_n + b_n}{2}}{4} = \dfrac{6a_{n+1} + a_n + b_n}{8}$

or $\quad 4a_{n+1} = 3a_n + b_n$ donc $b_n = 4a_{n+1} - 3a_n$

donc $a_{n+2} = \dfrac{6a_{n+1} + a_n + 4a_{n+1} - 3a_n}{8} = \dfrac{10}{8} a_{n+1} - \dfrac{2}{8} a_n$

$$a_{n+2} = \frac{5}{4} a_{n+1} - \frac{1}{4} a_n$$

(b)

Équation caractéristique : $x^2 - \frac{5}{4}x + \frac{1}{4} = 0 \iff 4x^2 - 5x + 1 = 0$

$\iff x = 1$ ou $x = \frac{1}{4}$

Donc $\exists \lambda, \mu \in \mathbb{R} / \forall n \in \mathbb{N}, \; a_n = \lambda + \mu \left(\frac{1}{4}\right)^n$, avec :

$$\begin{cases} \lambda + \mu = a_0 = 0 \\ \lambda + \frac{1}{4}\mu = a_1 = \frac{3a_0 + b_0}{4} = \frac{b_0}{4} = \frac{1}{4} \end{cases}$$

$\underset{L_2 \leftarrow L_2 - L_1}{\iff} \begin{cases} \lambda + \mu = 0 \\ -\frac{3}{4}\mu = \frac{1}{4} \end{cases} \iff \begin{cases} \lambda = -\mu = \frac{1}{3}. \\ \mu = -\frac{1}{3} \end{cases}$

Conclusion : $\boxed{\forall n \in \mathbb{N}, \; a_n = \frac{1}{3} - \frac{1}{3} \times \left(\frac{1}{4}\right)^n}$

$\forall n \in \mathbb{N}, \; b_n = 4a_{n+1} - 3a_n = \frac{4}{3} - \frac{4}{3} \times \left(\frac{1}{4}\right)^{n+1} - 1 - \left(\frac{1}{4}\right)^n$

$= \frac{1}{3} - \frac{4}{3} \times \frac{1}{4} \times \left(\frac{1}{4}\right)^n - \left(\frac{1}{4}\right)^n = \frac{1}{3} - \left(\frac{1}{4}\right)^n \left(\frac{1}{3} + 1\right).$

$$\boxed{b_n = \frac{1}{3} - \frac{4}{3} \times \left(\frac{1}{4}\right)^n}$$

② (a) $\forall n\in\mathbb{N},\quad 2a_{n+1}+b_{n+1}=\cancel{2}\times\dfrac{3a_n+b_n}{\cancel{4}2}+\dfrac{a_n+b_n}{2}$

$$=\frac{4a_n+2b_n}{2}=2a_n+b_n$$

Donc $(2a_n+b_n)$ est constante

(b) $\forall n\in\mathbb{N},\quad 2a_n+b_n=2\times a_0+b_0=1.$ donc $b_n=1-2a_n$

donc $\forall n\in\mathbb{N},\quad a_{n+1}=\dfrac{3a_n+1-2a_n}{4}=\dfrac{a_n+1}{4}$

$$a_{n+1}=\frac{1}{4}a_n+\frac{1}{4}$$

(c) On résout : $x=\frac{1}{4}x+\frac{1}{4}\iff 4x=x+1\iff x=\frac{1}{3}.$

Donc $\left(a_n-\frac{1}{3}\right)_n$ est géométrique de raison $\frac{1}{4}$:

$\forall n\in\mathbb{N},\quad a_n-\frac{1}{3}=\left(\frac{1}{4}\right)^n\left(a_0-\frac{1}{3}\right)=\left(\frac{1}{4}\right)^n\times\left(-\frac{1}{3}\right)$

donc : $\boxed{\forall n\in\mathbb{N},\quad a_n=\frac{1}{3}-\frac{1}{3}\times\left(\frac{1}{4}\right)^n}$

et on recommence : $\forall n\in\mathbb{N},\ b_n=4a_{n+1}-3a_n$ ... etc..

Exercice 5

- $\sin(p-q)=\sin(p)\cos(q)-\cos(p)\sin(q)$

$$=\cos(q)\times\cos(p)\times\frac{\sin(p)}{\cos(p)}-\cos(p)\times\cos(q)\times\frac{\sin(q)}{\cos(q)}$$

$$\sin(p-q)=\cos(p)\cos(q)\left(\tan(p)-\tan(q)\right)$$

① On a :

$$\underbrace{\sin\left((k+1)\alpha - k\alpha\right)} = \cos\left((k+1)\alpha\right)\cos(k\alpha)\left(\tan((k+1)\alpha) - \tan(k\alpha)\right)$$

$\sin(\alpha) \neq 0$ donc :

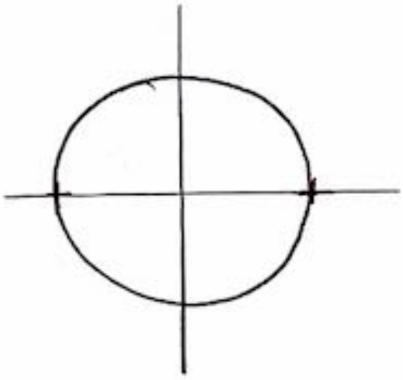
$$\frac{1}{\cos((k+1)\alpha)\cos(k\alpha)} = \frac{\tan((k+1)\alpha) - \tan(k\alpha)}{\sin(\alpha)}$$

donc $S_n = \frac{1}{\sin\alpha} \sum_{k=0}^{n} \left(\tan((k+1)\alpha) - \tan(k\alpha)\right)$ (télescopage)

$$= \frac{1}{\sin\alpha}\left(\tan((n+1)\alpha) - \underbrace{\tan(0)}_{=0}\right)$$

soit $\underline{S_n = \frac{\tan((n+1)\alpha)}{\sin(\alpha)}}$

② $\sin(\alpha) = 0 \iff \begin{cases} \alpha = 2\ell\pi,\ \ell \in \mathbb{Z} \\ \text{ou } \alpha = \pi + 2\ell\pi,\ \ell \in \mathbb{Z} \end{cases}$



• <u>si $\alpha = 2\ell\pi,\ \ell \in \mathbb{Z}$</u>

$\cos(k\alpha) = \cos(2(k\ell)\pi) = \cos(0) = 1.$

$\cos((k+1)\alpha) = \cos(\alpha + 2(k\ell\pi)) = \cos(\alpha) = \cos(2\ell\pi) = \cos(0) = 1$

Donc $\boxed{S_n = n+1}$

• <u>si $\alpha = \pi + 2\ell\pi,\ \ell \in \mathbb{Z}$</u>

$\cos(k\alpha) = \cos(k\pi + 2(k\ell)\pi) = \cos(k\pi) = \begin{cases} 1 & \text{si } k \text{ pair} \\ -1 & \text{si } k \text{ impair.} \end{cases} = (-1)^k$

$\cos((k+1)\alpha) = \cos(\alpha + k\alpha)$

$= \cos(\pi + 2\ell\pi + k\pi + 2(k\ell)\pi) = \cos((k+1)\pi) = (-1)^{k+1}$

Donc $S_n = \sum_{k=0}^{n} \frac{1}{\underbrace{(-1)^k(-1)^{k+1}}_{(-1)^{2k+1} = (-1)^{2k} \times (-1) = -1}}$ donc $\boxed{S_n = -(n+1)}$

(3) . Si $\alpha = k\pi$, $k \in \mathbb{Z}$: $\sin(\alpha) = 0$ donc $S_n = \pm(n+1) \neq 0$.

. si $\alpha \neq k\pi$, $k \in \mathbb{Z}$: $\sin(\alpha) \neq 0$

$$S_n = \frac{\tan((n+1)\alpha)}{\sin(\alpha)} = 0 \iff \tan((n+1)\alpha) = 0$$
$$\iff (n+1)\alpha = \ell\pi, \ \ell \in \mathbb{Z}$$
$$\boxed{S_n = 0 \iff \alpha = \frac{\ell}{n+1}\pi, \ \ell \in \mathbb{Z}}$$

Rigueur : $\frac{\ell}{n+1} \in \mathbb{Z} \iff n+1$ divise ℓ ...

Exercice 7

(1) Par récurrence sur $p \in \mathbb{N}$:

$$\mathcal{P}(p) : \text{“} \forall n \in \mathbb{N}, \forall q \in \mathbb{N}, \sum_{k=0}^{n} \binom{p}{k}\binom{q}{n-k} = \binom{p+q}{n} \text{”}.$$

* $p=0$: $\sum_{k=0}^{n} \binom{0}{k}\binom{q}{n-k} = \binom{0}{0}\binom{q}{n} = \binom{q}{n}$ et $\binom{0+q}{n} = \binom{q}{n}$ ✓

$\binom{0}{k} = 0$ si $k \geq 1$

* $p \geq 0$: Supposons $\mathcal{P}(p)$ à un certain rang p.

Montrons que, $\sum_{k=0}^{n} \binom{p+1}{k}\binom{q}{n-k} = \binom{p+1+q}{n}$

$$\sum_{k=0}^{n} \underbrace{\binom{p+1}{k}}_{= \binom{p}{k} + \binom{p}{k-1}}\binom{q}{n-k} = \underbrace{\sum_{k=0}^{n} \binom{p}{k}\binom{q}{n-k}}_{(HR)} + \sum_{k=1}^{n} \underbrace{\binom{p}{k-1}}_{=0 \text{ si } k-1<0}\binom{q}{n-k}$$

(triangle de Pascal)

Donc : $\displaystyle\sum_{k=0}^{n}\binom{p+1}{k}\binom{q}{n-k}=\binom{p+q}{n}+\sum_{k=0}^{n-1}\binom{p}{k}\binom{q}{n-(k+1)}$ $(k'=k-1)$

$$=\binom{p+q}{n}+\underbrace{\sum_{k=0}^{n-1}\binom{p}{k}\binom{q}{(n-1)-k}}_{(HR)}$$

$$=\binom{p+q}{n}+\binom{p+q}{n-1}$$

$$\underset{\text{(triangle de Pascal)}}{=}\binom{p+q+1}{n} \quad \checkmark \qquad \text{récurrence achevée.}$$

(2) $\displaystyle\sum_{k=0}^{n}\binom{n}{k}^2=\sum_{k=0}^{n}\binom{n}{k}\times\binom{n}{k}\underset{\text{Symétrie}}{=}\sum_{k=0}^{n}\binom{n}{k}\times\binom{n}{n-k}$

$$\underset{(1)}{=}\binom{n+n}{n}=\binom{2n}{n}$$

(3) (a) $\forall n\in\mathbb{N}$, $\displaystyle T_n=\sum_{k=\cancel{0}/1}^{n}k\binom{n}{k}^2\underset{\text{def}}{=}n\sum_{k=1}^{n}\binom{n-1}{k-1}\binom{n}{k}$

$$\underset{\text{Symétrie}}{=}n\sum_{k=1}^{n}\binom{n-1}{(n-1)-(k-1)}\binom{n}{k}$$

$$=n\sum_{k=\cancel{1}\,0}^{n}\binom{n-1}{n-k}\binom{n}{k}\underset{(1)}{=}n\binom{n-1+n}{n}$$

car $\binom{n-1}{n}=0$.

Donc $T_n=n\binom{2n-1}{n}$

(b) $\forall n\in\mathbb{N}$, $\displaystyle T_n=n\times\frac{(2n-1)!}{n!\,(2n-1-n)!}=n\frac{(2n-1)!}{n!\,(n-1)!}=\frac{(2n)\,(2n-1)!}{2\,n!\,(n-1)!}$

$$=\frac{(2n)!}{2\,n!\,(n!)}\times n=\frac{n}{2}\,\frac{(2n)!}{n!\,(2n-n)!}=\frac{n}{2}\binom{2n}{n}$$