DM3 - lundi 27 Novembre
Corrigé

Exercice 1

(1) <u>Ensemble de résolution :</u> $\frac{x-2}{x^2-1} > 0$ et $x^2-1 \neq 0$.

x	$-\infty$		-1		1		2		$+\infty$
$x-2$		$-$	|	$-$	|	$-$	0	$+$	
x^2-1		$+$	0	$-$	0	$+$	|	$+$	
$\frac{x-2}{x^2-1}$		$-$	||	$+$	||	$-$	0	$+$	

Donc <u>on résout sur $]-1,1[\cup]2,+\infty[$:</u>

(1) $\Leftrightarrow \frac{x-2}{x^2-1} \geqslant 1 \Leftrightarrow \frac{x-2}{x^2-1} - 1 \geqslant 0 \Leftrightarrow \frac{x-2-x^2+1}{x^2-1} \geqslant 0$

$\Leftrightarrow \frac{-x^2+x-1}{x^2-1} \geqslant 0$

$\Leftrightarrow \frac{x^2-x+1}{x^2-1} \leq 0.$

or $x^2-x+1 > 0 \ \forall x \in \mathbb{R}$ car $\Delta < 0$.

<u>Conclusion :</u> (1) $\Leftrightarrow x^2-1 < 0$

$\boxed{\mathcal{S} =]-1,1[}$

(2) <u>On résout sur $]0,+\infty[$:</u>

(2) $\Leftrightarrow \left(\frac{1}{2}\ln(x)\right)^2 - \ln(2) - \ln(x) + 2\ln(2) > 0$

$\Leftrightarrow \frac{1}{4}\ln^2(x) - \ln(x) + \ln(2) > 0 \Leftrightarrow (\ln(x))^2 - 4\ln(x) + 4\ln(2) > 0$

Posons $X = \ln(x) \in \mathbb{R}$.

donc (2) $\iff X^2 - 4X + 4\ln(2) > 0$

$$\Delta = 16 - 16\ln(2) = 16\left(1 - \ln(2)\right)$$

or $2 < e$ donc $\ln(2) < \ln(e)$

soit $\ln(2) < 1$ donc $\Delta > 0$

donc

X	$-\infty$		$2-2\sqrt{1-\ln(2)}$		$2+2\sqrt{1-\ln(2)}$		$+\infty$
$X^2-4X+4\ln(2)$		$+$	0	$-$	0	$+$	

$$(2) \iff X < 2 - 2\sqrt{1-\ln(2)} \text{ ou } X > 2 + 2\sqrt{1-\ln(2)}$$

$$\iff \ln(x) < 2\left(1-\sqrt{1-\ln(2)}\right) \text{ ou } \ln(x) > 2\left(1+\sqrt{1-\ln(2)}\right)$$

$$\iff x < e^{2\left(1-\sqrt{1-\ln(2)}\right)} \text{ ou } x > e^{2\left(1+\sqrt{1-\ln(2)}\right)}$$

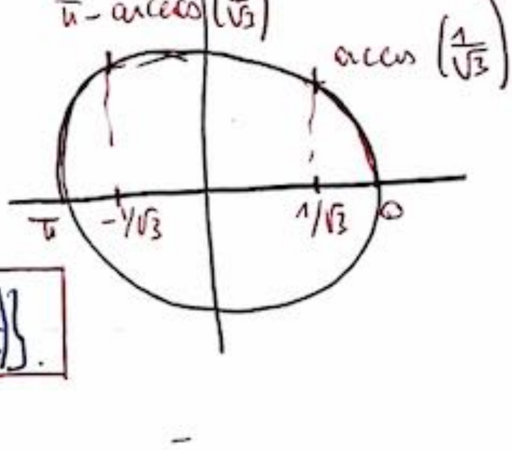
$$\boxed{S = \left]0, e^{2\left(1-\sqrt{1-\ln(2)}\right)}\right[\cup \left]e^{2\left(1+\sqrt{1-\ln(2)}\right)}, +\infty\right[}$$

Exercice 2

(1) $0 \leq x \leq 2\pi$ donc $0 \leq \frac{x}{2} \leq \pi$

(1) $\iff \cos\left(\frac{x}{2}\right) = \frac{1}{\sqrt{3}}$ ou $\cos\left(\frac{x}{2}\right) = -\frac{1}{\sqrt{3}}$

$\iff \frac{x}{2} = \arccos\left(\frac{1}{\sqrt{3}}\right)$ ou $\frac{x}{2} = \pi - \arccos\left(\frac{1}{\sqrt{3}}\right)$



donc $\boxed{S = \left\{2\arccos\left(\frac{1}{\sqrt{3}}\right), 2\pi - 2\arccos\left(\frac{1}{\sqrt{3}}\right)\right\}.}$

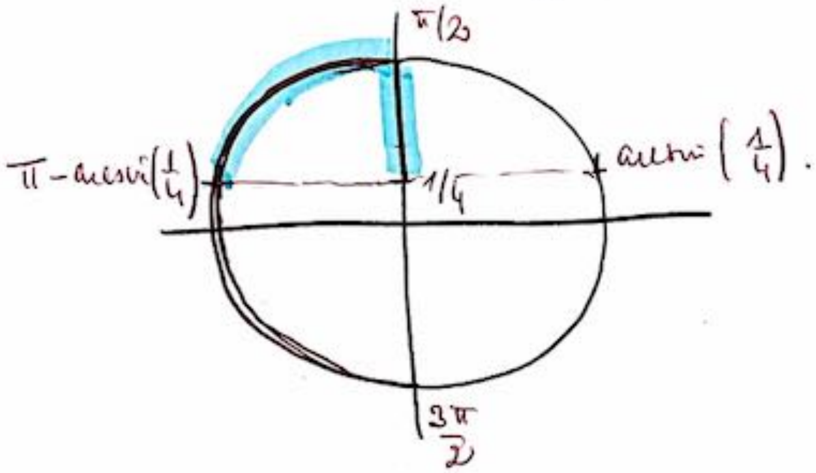
$\arccos(\theta) \in [0, \pi]$

donc $2\arccos(\theta) \in [0, 2\pi]$ ✓ et $-2\arccos(\theta) \in [-2\pi, 0]$

donc $2\pi - 2\arccos(\theta) \in [0, 2\pi]$ ✓

(2) $-\pi \leq x \leq \pi$

$-\frac{\pi}{2} \leq \frac{x}{2} \leq \frac{\pi}{2}$ donc $\frac{\pi}{2} \leq \frac{x}{2} + \pi \leq \frac{3\pi}{2}$



donc (2) $\Leftrightarrow$ $\frac{\pi}{2} \leq \frac{x}{2} + \pi < \pi - \arcsin\left(\frac{1}{4}\right)$

$\Leftrightarrow$ $-\frac{\pi}{2} \leq \frac{x}{2} < -\arcsin\left(\frac{1}{4}\right)$

donc $\boxed{S = \left[-\pi, -2\arcsin\left(\frac{1}{4}\right)\right[}$

$\arcsin(\theta) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ donc $2\arcsin(\theta) \in [-\pi, \pi]$

donc $-2\arcsin(\theta) \in [-\pi, \pi]$ ✓

Exercice 3

(1)

$n \backslash p$	0	1	2	3	4	
0	1					
1	1	1				
2	1	2	1			
3	1	3	3	1		
→ 4	1	4	6	4	1	

Binôme de Newton:

$$(1-3i)^4 = 1\times 1^4\,(-3i)^0 + 4\times 1^3\times(-3i) + 6\times 1^2\times(-3i)^2 + 4\times 1\times(-3i)^3 + 1\times 1\times(-3i)^4$$

$$= 1 - 12i + 6\times(-9) + 4\times 27\,i + 81$$

$$= (1 - 54 + 81) + (-12 + 108)\,i = 28 + 96\,i$$

donc $\boxed{i\,(1-3i)^4 = -96 + 28\,i}$

(2) $$z_2 = \frac{(1-2i)\,(2-\sqrt{3}\,i)}{4+3} = \frac{(2-2\sqrt{3}) + (-4-\sqrt{3})\,i}{7}$$

$$\boxed{z_2 = \frac{2}{7}(1-\sqrt{3}) - \frac{(4+\sqrt{3})}{7}\,i}$$

(3) $$z_3 = \frac{\overline{(3+5i)^2}}{1+2i} = \frac{\left(\overline{3+5i}\right)^2}{1+2i} = \frac{(3-5i)^2}{1+2i} = \frac{(3-5i)^2\,(1-2i)}{1+4}$$

$$= \frac{1}{5}\left[(-16, -30i)\,(1-2i)\right]$$

$$= -\frac{2}{5}\,(8+15i)\cdot(1-2i) = -\frac{2}{5}\,(25-i)$$

donc $\boxed{z_3 = -10 + \frac{2}{5}\,i}$

Exercice 4

1. $$(1+i)(1+2i)(1+3i) = \left[(1-2)+(1+2)i\right](1+3i)$$
$$= (-1+3i)(1+3i)$$
$$= -(\overline{1+3i})(1+3i)$$
$$= -|1+3i|^2 = -(1+9)$$

donc $\boxed{(1+i)(1+2i)(1+3i) = -10.}$

2. (a) $1+i = \sqrt{2}\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)$ donc $\boxed{1+i = \sqrt{2}\, e^{i\frac{\pi}{4}}}$

(b) $1+2i = \sqrt{5}\left(\frac{1}{\sqrt{5}} + \frac{2}{\sqrt{5}} i\right)$

on cherche $\theta \in [-\pi, \pi[$ tq $\cos(\theta) = \frac{1}{\sqrt{5}}$ et $\sin(\theta) = \frac{2}{\sqrt{5}}$

donc $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = 2$

on prend $\theta = \arctan(2)$. θ convient, en effet :

- $\sin(\theta) = \tan(\theta)\cos(\theta) = 2 \times \cos(\arctan(2)) = 2 \times \frac{1}{\sqrt{5}} = \frac{2}{\sqrt{5}}$
- $\cos(\theta) = \frac{1}{\sqrt{5}}$
- $\theta = \arctan(2) \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\subset [-\pi, \pi[$

Exo 15 :
$\cos^2(\theta) = \frac{1}{1+\tan^2(\theta)}$

donc $|\cos(\theta)| = \cos(\theta) = \frac{1}{\sqrt{1+\tan^2(\theta)}}$

car $\theta \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$

$= \frac{1}{\sqrt{1+2^2}}$

Conclusion : $\boxed{1+2i = \sqrt{5}\, e^{i\arctan(2)}}$

(c) $1+3i = \sqrt{10}\left(\frac{1}{\sqrt{10}} + \frac{3}{\sqrt{10}} i\right)$

Prenons $\theta = \arctan(3)$:
$\theta \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\subset [-\pi, \pi[$

- $\cos(\theta) = \frac{1}{\sqrt{1+\tan^2(\theta)}} = \frac{1}{\sqrt{1+3^2}} = \frac{1}{\sqrt{10}}$.
- $\sin(\theta) = \tan(\theta)\cos(\theta) = \frac{3}{\sqrt{10}}$.

Donc $\boxed{1+3i = \sqrt{10}\, e^{i\arctan(3)}}$

(d) $(1+i)(1+2i)(1+3i) = \sqrt{2}\, e^{i\pi/4} \times \sqrt{5}\, e^{i\arctan(2)} \times \sqrt{10}\, e^{i\arctan(3)}$

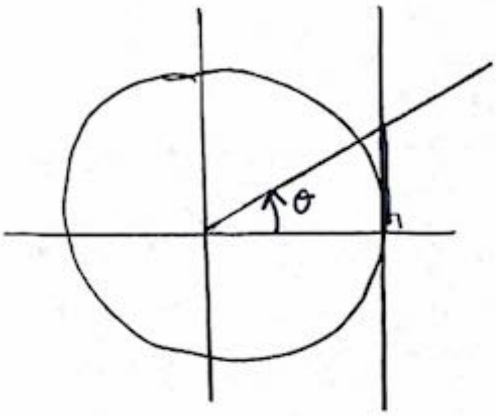
$$= \sqrt{2\times5\times10}\,.\, e^{i\left(\pi/4+\arctan(2)+\arctan(3)\right)}$$

donc $\boxed{(1+i)(1+2i)(1+3i) = 10\, e^{i\left(\arctan(1)+\arctan(2)+\arctan(3)\right)}}$

(3) (a) $\forall x \in \mathbb{R},\ -\frac{\pi}{2} < \arctan(x) < \frac{\pi}{2}$

donc $-\frac{3\pi}{2} < \arctan(1)+\arctan(2)+\arctan(3) < \frac{3\pi}{2}$.

or $1, 2$ et 3 sont positifs donc $\begin{cases} \arctan(1) \geq 0 \\ \arctan(2) \geq 0 \\ \arctan(3) \geq 0. \end{cases}$



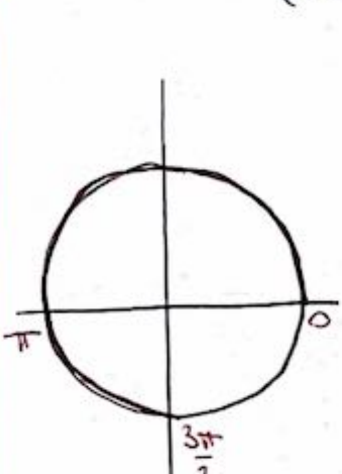
Conclusion: $\boxed{\arctan(1)+\arctan(2)+\arctan(3) \in \left[0, \frac{3\pi}{2}\right[}$

(b) On a:

$(1+i)(1+2i)(1+3i) = -10 = 10\, e^{i\pi} = 10\, e^{i\left(\arctan(1)+\arctan(2)+\arctan(3)\right)}$

donc $\arctan(1)+\arctan(2)+\arctan(3) = \pi + 2k\pi,\ k \in \mathbb{Z}$

Or $\arctan(1)+\arctan(2)+\arctan(3) \in \left[0, \frac{3\pi}{2}\right[$



donc $\boxed{\arctan(1)+\arctan(2)+\arctan(3) = \pi}$