DM4 - Corrigé
Vendredi 8 Décembre

Exercice 1

$$1+i = \sqrt{2}\, e^{i\frac{\pi}{4}} \quad \text{donc } (1+i)^{2000} = \left(\sqrt{2}\, e^{i\frac{\pi}{4}}\right)^{2000}$$

$$= \left(\sqrt{2}\right)^{2000} \left(e^{i\frac{\pi}{4}}\right)^{2000}$$

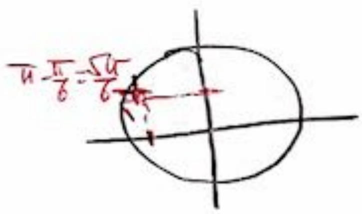
$$= 2^{1000}\, e^{i\cdot\frac{2000\pi}{4}}$$

$$= 2^{1000}\, e^{i\cdot 500\pi}$$

or $500\pi = 2\times(250)\pi$ donc $e^{i500\pi} = e^{i\cdot(250)\times 2\pi} = e^{i0} = 1$

donc $1+i = 2^{1000}$

$$i-\sqrt{3} = 2\left(-\frac{\sqrt{3}}{2} + i\frac{1}{2}\right) = 2e^{i\frac{5\pi}{6}}$$



$$\text{donc } \left(i-\sqrt{3}\right)^{1000} = 2^{1000}\, e^{i\cdot\frac{5\times 1000\pi}{6}}$$

$$\text{or } \frac{5\times 1000\pi}{6} = \frac{5\times 500\pi}{3} = \frac{5\times(166\times 3+2)\pi}{3} = (5\times 166)\pi + \frac{10\pi}{3}$$

$$= (5\times 83)\times 2\pi + 3\pi + \frac{\pi}{3}.$$

$$= (5\times 83+1)\, 2\pi + \frac{4\pi}{3}$$

$$\text{donc } e^{i\frac{5\times 1000}{6}\pi} = e^{i\frac{4\pi}{3}}$$

$$\text{donc } \left(i-\sqrt{3}\right)^{1000} = 2^{1000}\, e^{i\frac{4\pi}{3}}$$

$$\text{Donc : } z = \frac{2^{1000}}{2^{1000}\, e^{i\frac{4\pi}{3}}} = e^{-i\frac{4\pi}{3}} = e^{i\frac{2\pi}{3}} = j \text{, soit } \boxed{z = -\frac{1}{2} + i\frac{\sqrt{3}}{2}}$$

Exercice 2

$$1-e^{i\frac{\pi}{3}} = e^{i0} - e^{i\frac{\pi}{3}} = e^{i\frac{\pi}{6}}\left(e^{-i\frac{\pi}{6}} - e^{i\frac{\pi}{6}}\right)$$

$$= -e^{i\frac{\pi}{6}}\left(e^{i\frac{\pi}{6}} - e^{-i\frac{\pi}{6}}\right)$$

$$= -2i\sin\left(\frac{\pi}{6}\right)e^{i\frac{\pi}{6}}.$$

de même : $1+e^{i\frac{\pi}{3}} = 2\cos\left(\frac{\pi}{6}\right)e^{i\frac{\pi}{6}}$

donc : $z = \dfrac{-\cancel{2}i\sin\left(\frac{\pi}{6}\right)\cancel{e^{i\pi/6}}}{\cancel{2}\cos\left(\frac{\pi}{6}\right)\cancel{e^{i\pi/6}}} = -i\tan\left(\frac{\pi}{6}\right) = i\tan\left(\frac{\pi}{6}\right)e^{i\pi}$

or $i = e^{i\frac{\pi}{2}}$ donc $z = \tan\left(\frac{\pi}{6}\right)e^{i\left(\pi+\frac{\pi}{2}\right)} = \tan\left(\frac{\pi}{6}\right)e^{i\frac{3\pi}{2}}$

or $\tan\left(\frac{\pi}{6}\right) > 0$ donc $\boxed{z = \tan\left(\frac{\pi}{6}\right)e^{i\frac{3\pi}{2}}}$

Exercice 3

1. • $\sqrt{3}+i = 2e^{i\frac{\pi}{6}}$ donc :

$$z^2 = \sqrt{3}+i \iff z^2 = 2e^{i\frac{\pi}{6}} \iff z^2 - 2e^{i\frac{\pi}{6}} = 0$$

$$\iff z^2 - \left(\sqrt{2}e^{i\frac{\pi}{12}}\right)^2 = 0$$

$$\iff z = \sqrt{2}e^{i\frac{\pi}{12}} \text{ ou } z = -\sqrt{2}e^{i\frac{\pi}{12}} = \sqrt{2}e^{i\left(\frac{\pi}{12}+\pi\right)}$$

donc $\boxed{S = \left\{\sqrt{2}\left(\cos\left(\frac{\pi}{12}\right)+i\sin\left(\frac{\pi}{12}\right)\right) ; \sqrt{2}\left(\cos\left(\frac{13\pi}{12}\right)+i\sin\left(\frac{13\pi}{12}\right)\right)\right\}}$.

- On pose $z = x + iy$, $(x, y) \in \mathbb{R}^2$

alors $z^2 = \sqrt{3} + i \iff (x+iy)^2 = \sqrt{3} + i$

$$\iff (x^2 - y^2) + i(2xy) = \sqrt{3} + i$$

$$\iff \begin{cases} x^2 - y^2 = \sqrt{3} \\ 2xy = 1 \end{cases}$$

or $|z|^2 = x^2 + y^2 = |\sqrt{3} + i| = 2$ donc : $\begin{cases} x^2 - y^2 = \sqrt{3} & L_1 \\ x^2 + y^2 = 2 & L_2 \end{cases}$

$$\iff \begin{cases} 2x^2 = 2 + \sqrt{3} & L_1 \leftarrow L_1 + L_2 \\ y^2 = 2 - x^2 \end{cases}$$

$$\iff \begin{cases} x = \sqrt{\frac{2+\sqrt{3}}{2}} \\ y^2 = 2 - x^2 = \frac{2-\sqrt{3}}{2} \end{cases} \text{ ou } \begin{cases} x = -\sqrt{\frac{2+\sqrt{3}}{2}} \\ y^2 = 2 - x^2 = \frac{2-\sqrt{3}}{2} \end{cases}$$

$$\iff \begin{cases} x = \sqrt{\frac{2+\sqrt{3}}{2}} \\ y = \sqrt{\frac{2-\sqrt{3}}{2}} \end{cases} \text{ ou } \begin{cases} x = \sqrt{\frac{2+\sqrt{3}}{2}} \\ y = -\sqrt{\frac{2-\sqrt{3}}{2}} \end{cases} \text{ ou } \begin{cases} x = -\sqrt{\frac{2+\sqrt{3}}{2}} \\ y = \sqrt{\frac{2-\sqrt{3}}{2}} \end{cases} \text{ ou } \begin{cases} x = -\sqrt{\frac{2+\sqrt{3}}{2}} \\ y = -\sqrt{\frac{2-\sqrt{3}}{2}} \end{cases}$$

or $\underbrace{xy > 0}_{(x \text{ et } y \text{ de même signe})}$ donc : $\boxed{S = \left\{ \sqrt{\frac{2+\sqrt{3}}{2}} + i\sqrt{\frac{2-\sqrt{3}}{2}} \; ; \; -\sqrt{\frac{2+\sqrt{3}}{2}} - i\sqrt{\frac{2-\sqrt{3}}{2}} \right\}}$

2. On a sous forme trigonométrique → algébrique :

$\underbrace{\sqrt{2}\cos\left(\frac{\pi}{12}\right)}_{>0 \text{ car } \frac{\pi}{12} \in \left[-\frac{\pi}{2}; \frac{\pi}{2}\right]} + i\sqrt{2}\sin\left(\frac{\pi}{12}\right)$ et $\underbrace{\sqrt{2}\cos\left(\frac{13\pi}{12}\right)}_{<0 \text{ car } \frac{13\pi}{12} \in \left[\frac{\pi}{2}; \frac{3\pi}{2}\right]} + i\sqrt{2}\sin\left(\frac{13\pi}{12}\right)$.

donc : $\sqrt{2}\cos\left(\frac{\pi}{12}\right) + i\sqrt{2}\sin\left(\frac{\pi}{12}\right) = \sqrt{\frac{2+\sqrt{3}}{2}} + i\sqrt{\frac{2-\sqrt{3}}{2}}$

donc $\boxed{\cos\left(\frac{\pi}{12}\right) = \frac{\sqrt{2+\sqrt{3}}}{2}}$