DM3 _ Corrigé
4/10/2024

Exercice 1

(1.)

- $\frac{1}{2}(1+\sqrt{5})^2 = \frac{1+2\sqrt{5}+5}{2} = \frac{6+2\sqrt{5}}{2} = \frac{\cancel{2}(3+\sqrt{5})}{\cancel{2}} = 3+\sqrt{5}$ ✓

- $\frac{1}{2}(1-\sqrt{5})^2 = \frac{1-2\sqrt{5}+5}{2} = \frac{6-2\sqrt{5}}{2} = 3-\sqrt{5}$ ✓

(2.) Par récurrence double sur $n \in \mathbb{N}$:

$$u_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

* $\underline{n=0}$ $\quad u_0 = 0$

et $\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^0 - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^0 = \frac{1}{\sqrt{5}} - \frac{1}{\sqrt{5}} = 0$ ✓

* $\underline{n=1}$: $\quad u_1 = 1$

et $\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^1 - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^1 = \frac{\cancel{1}+\sqrt{5}-\cancel{1}+\sqrt{5}}{2\sqrt{5}} = \frac{2\sqrt{5}}{2\sqrt{5}} = 1$ ✓

Récurrence initialisée

- Sq $u_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$ et $u_{n+1} = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}$

à un certain rang n.

Mq $u_{n+2} = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+2} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+2}$

$$u_{n+2} = u_{n+1} + u_n$$

$$\underset{(HR)}{=} \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1} + \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n}$$

$$= \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n}\left[\frac{1+\sqrt{5}}{2}+1\right] - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n}\left[\left(\frac{1-\sqrt{5}}{2}\right)+1\right].$$

$$= \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n}\left(\frac{3+\sqrt{5}}{2}\right) - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n}\left(\frac{3-\sqrt{5}}{2}\right)$$

$$\underset{q^{\circ}(1)}{=} \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n}\left(\frac{1+\sqrt{5}}{2}\right)^{2} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n}\left(\frac{1-\sqrt{5}}{2}\right)^{2}$$

$$= \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+2} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+2} \quad \checkmark$$

Conclusion: par récurrence double, $\forall n \in \mathbb{N}$, $u_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n}$

Exercice 21

1. $\forall n \geq 2$,

$$\ln\left(1-\frac{1}{n^2}\right) = \ln\left(\frac{n^2-1}{n^2}\right) = \ln(n^2-1) - \ln(n^2)$$

$$= \ln\left((n-1)(n+1)\right) - 2\ln(n)$$

$$= \ln(n+1) + \ln(n-1) - 2\ln(n) \quad \checkmark$$

(2.) $\forall n \geq 2,\quad S_n = \sum_{k=2}^{n} \ln\left(1 - \frac{1}{k^2}\right)$

$$= \underset{q^\circ(1)}{=} \sum_{k=2}^{n} \ln(k+1) + \sum_{k=2}^{n} \ln(k-1) - 2\sum_{k=1}^{n} \ln(k)$$

$$\underset{\substack{k'=k+1\\ j=k-1}}{=} \sum_{k'=3}^{n+1} \ln(k') + \sum_{j=1}^{n-1} \ln(j) - 2\sum_{k=1}^{n} \ln(k)$$

$$= \ln(n) + \ln(n+1) + \sum_{k=3}^{n-1} \ln(k) + \cancel{\ln(1)} + \ln(2) + \sum_{k=3}^{n-1} \ln(k)$$
$$- 2\ln(2) - 2\cancel{\ln(1)} - 2\ln(n) - 2\sum_{k=3}^{n-1} \ln(k)$$

$$S_n = \ln(n+1) - \ln(n) - \ln(2) + \cancel{(1+1-2)} \sum_{k=3}^{n-1} \ln(k)$$

$$\boxed{S_n = \ln\left(\frac{n+1}{2n}\right)}$$

Exercice 3

(1.) $S_1 = \frac{1}{2n} \sum_{k=0}^{n-1} \left(\sqrt{n}\,k - 1\right) = \frac{1}{2n}\left(\sqrt{n} \sum_{k=0}^{n-1} k - n\right)$

$$= \frac{1}{2n}\left(\sqrt{n}\,\frac{(n-1)\times n}{2} - n\right) = \frac{\cancel{n}}{2\cancel{n}}\left(\frac{(n-1)\sqrt{n}}{2} - 1\right)$$

$$= \frac{(n-1)\sqrt{n} - 2}{2}$$

$$\boxed{S_1 = \frac{n\sqrt{n} - \sqrt{n} - 2}{4}}$$

2.

$$S_2 = \sum_{k=2}^{n} \frac{k^2(k-1)}{\sqrt{5}} = \frac{1}{\sqrt{5}} \sum_{k=2}^{n} (k^3 - k^2)$$

$$= \frac{1}{\sqrt{5}}\left(\sum_{k=2}^{n} k^3 - \sum_{k=2}^{n} k^2 \right)$$

$$= \frac{1}{\sqrt{5}}\left(\sum_{k=1}^{n} k^3 \cancel{-1^3} - \sum_{k=1}^{n} k^2 \cancel{+1^2} \right)$$

$$= \frac{1}{\sqrt{5}}\left(\frac{n^2(n+1)^2}{4} - \frac{n(n+1)(2n+1)}{6} \right)$$

$$= \frac{n(n+1)}{\sqrt{5}}\left(\frac{n(n+1)}{4} - \frac{2n+1}{6} \right)$$

$$= \frac{n(n+1)}{\sqrt{5}}\left(\frac{3n^2+3n-4n-2}{12} \right)$$

$$\boxed{S_2 = \frac{n(n+1)(3n^2-n-2)}{12\sqrt{5}}}$$

3.

$$S_3 = \sum_{k=4}^{2n+1} \left(\frac{1}{\sqrt{2}}\right)^{2n-k} \times (\sqrt{5})^{2k+1}$$

$$= \left(\frac{1}{\sqrt{2}}\right)^{2n} \times \sqrt{5} \times \sum_{k=4}^{2n+1} \left(\frac{1}{\sqrt{2}}\right)^{-k} \times (\sqrt{5})^{2k}$$

$$(\sqrt{5})^{2k} = \left[(\sqrt{5})^2\right]^k = 5^k \qquad \left(\frac{1}{\sqrt{2}}\right)^{-k} = (\sqrt{2})^k$$

$$= \left(\frac{1}{2}\right)^{n} \times \sqrt{5} \times \sum_{k=4}^{2n+1} (5\sqrt{2})^k$$

$$\underset{5\sqrt{2} \neq 1}{=} \frac{\sqrt{5}}{2^n} \times \frac{(5\sqrt{2})^4 - (5\sqrt{2})^{2n+2}}{1-5\sqrt{2}} = \frac{\sqrt{5}}{2^n} \, \frac{\left((5\sqrt{2})^4 - (5\sqrt{2})^{2n+2}\right)(1+5\sqrt{2})}{1-(5\sqrt{2})^2}$$

$$\boxed{S_3 = \frac{\sqrt{5}(1+5\sqrt{2})}{2^n \times 49} \left((5\sqrt{2})^{2n+2} - (5\sqrt{2})^4 \right)}$$