DM6. Corrigé
Vendredi 6 Décembre 2024.

Exercice 1

1. $$1+\cos(\theta)+i\sin(\theta) = 1+e^{i\theta} = e^{i\frac{\theta}{2}}\left(e^{-i\frac{\theta}{2}}+e^{i\frac{\theta}{2}}\right) = 2\cos\left(\frac{\theta}{2}\right)e^{i\frac{\theta}{2}}$$

donc $$(1+\cos\theta+i\sin\theta)^n = 2^n\cos^n\left(\frac{\theta}{2}\right)e^{in\frac{\theta}{2}}$$

$$= \boxed{2^n\cos^n\left(\frac{\theta}{2}\right)\cos\left(n\frac{\theta}{2}\right)+i\,2^n\cos^n\left(\frac{\theta}{2}\right)\sin\left(\frac{n\theta}{2}\right)}$$

2. $$-2ie^{ix} = 2i\,e^{i(x+\pi)} = 2\,e^{i\left(x+\pi+\frac{\pi}{2}\right)}$$

$2>0$ donc $$-2ie^{ix} = \boxed{2\left(\cos\left(x+\frac{3\pi}{2}\right)+i\sin\left(x+\frac{3\pi}{2}\right)\right)}$$

Exercice 2

- $$\sum_{k=1}^{n}\sin(2k+3) = \sum_{k=1}^{n}\operatorname{Im}\left(e^{i(2k+3)}\right) = \operatorname{Im}\left(\sum_{k=1}^{n}e^{i(2k+3)}\right)$$

- $$\sum_{k=1}^{n}e^{i(2k+3)} = e^{i3}\sum_{k=1}^{n}\left(e^{i2}\right)^k \underset{e^{2i}\neq 1}{=} e^{3i}\,\frac{e^{2i}-e^{2i(n+1)}}{1-e^{2i}}$$

$$= e^{5i}\,\frac{1-e^{2in}}{1-e^{2i}}$$

- $$1-e^{2in} = e^{in}\left(e^{-in}-e^{in}\right) = -2i\sin(n)\,e^{in}$$
- $$1-e^{2i} = e^{i}\left(e^{-i}-e^{i}\right) = -2i\sin(1)\,e^{i}$$

donc $\displaystyle\sum_{k=1}^{n} e^{i(2k+3)} = e^{5i} \frac{\cancel{-2i} \sin(n) e^{in}}{\cancel{-2i} \sin(1) e^{i}} = e^{5i} \frac{\sin(n)}{\sin(1)} e^{i(n-1)}$

$$= \frac{\sin(n)}{\sin(1)} e^{i(n+4)}$$

$$= \frac{\sin(n)}{\sin(1)} \cos(n+4) + i \frac{\sin(n)}{\sin(1)} \sin(n+4)$$

Donc $\boxed{\displaystyle\sum_{k=1}^{n} \sin(2k+3) = \frac{\sin(n)}{\sin(1)} \sin(n+4)}$

Exercice 3

1. (a) si k est pair : $k=2p$ donc $i^{2p} + (-i)^{2p} = (i^2)^p + i^{2p}$

$$= 2\times(i^2)^p = 2(-1)^p.$$

(b) si k est impair : $k=2p+1$ donc

$$i^k + (-i)^k = i^{2p+1} + (-i)^{2p+1}$$

$$= i^{2p+1} - i^{2p+1} = 0$$

2. $\displaystyle(2+3i)^n + (2-3i)^n = \sum_{k=0}^{n} \binom{n}{k} (3i)^k (2)^{n-k} + \sum_{k=0}^{n} \binom{n}{k} (-3i)^k 2^{n-k}$

$$= \sum_{k=0}^{n} \binom{n}{k} 2^{n-k} \left((3i)^k + (-3i)^k\right)$$

$$= \sum_{k=0}^{n} \binom{n}{k} 2^{n-k} 3^k \left(i^k + (-i)^k\right) \quad \checkmark$$

3. d'après (1)

$$(2+3i)^n + (2-3i)^n = \sum_{\substack{k=0 \\ \underline{k \text{ pair}}}}^{n} \underbrace{\binom{n}{k} 2^{n-k} 3^k}_{\in \mathbb{N}} \underbrace{\left(i^k + (-i)^k\right)}_{2(-1)^k \in \mathbb{N}} + 0 .$$

donc $(2+3i)^n + (2-3i)^n \in \mathbb{N}$. ✓