DS3 – Mathématiques – Corrigé – 13/11/2024

Exercice 2

(1.)
$$\sum_{k=1}^{n} \frac{\binom{n}{k+1}}{3^{2n-k}} \underset{k'=k+1}{=} \sum_{k=2}^{n+1} \frac{\binom{n}{k}}{3^{2n-(k-1)}} = \frac{1}{3^{n+1}} \sum_{k=2}^{n+1} \binom{n}{k}\left(\frac{1}{3}\right)^{n-k}$$

$$= \frac{1}{3^{n+1}} \left(\sum_{k=2}^{n} \binom{n}{k}\left(\frac{1}{3}\right)^{n-k} + \underbrace{\binom{n}{n+1}}_{=0}\left(\frac{1}{3}\right)^{n-(n+1)} \right)$$

$$= \frac{1}{3^{n+1}} \left(\sum_{k=0}^{n} \binom{n}{k}\left(\frac{1}{3}\right)^{n-k} - \binom{n}{0}\left(\frac{1}{3}\right)^{n-0} - \binom{n}{1}\left(\frac{1}{3}\right)^{n-1} \right)$$

$$= \frac{1}{3^{n+1}} \left(\left(\frac{1}{3}+1\right)^{n} - \left(\frac{1}{3}\right)^{n} - n\left(\frac{1}{3}\right)^{n-1} \right)$$

$$= \boxed{\frac{1}{3^{n+1}} \left(\left(\frac{4}{3}\right)^{n} - \left(\frac{1}{3}\right)^{n} - n\left(\frac{1}{3}\right)^{n-1} \right)}$$

0.5

(2.)
$$\sum_{1\leq j\leq k\leq n} \frac{jn}{k} = \sum_{k=1}^{n} \sum_{j=1}^{k} \frac{jn}{k}.$$

- $$\sum_{j=1}^{k} \frac{jn}{k} = \frac{n}{k} \sum_{j=1}^{k} j = \frac{n}{k} \times \frac{k(k+1)}{2} = \frac{n}{2}(k+1)$$
$$= \frac{n}{2}k + \frac{n}{2}$$

- $$\sum_{k=1}^{n} \left(\frac{n}{2}k + \frac{n}{2}\right) = \frac{n}{2} \sum_{k=1}^{n} k + \frac{n}{2} \times n = \frac{n}{2} \times \frac{n(n+1)}{2} + \frac{n^2}{2}$$

$$= \frac{n^2}{2}\left(\frac{n+1}{2}+1\right) = \frac{n^2}{2}\times\frac{n+3}{2} = \boxed{\frac{n^2(n+3)}{4}}$$

Exercice 3

(1.)

$$\binom{n+1}{k} = \frac{(n+1)!}{k!\,(n+1-k)!} \quad \text{et} \quad \frac{n+1}{k}\binom{n}{k-1} = \frac{n+1}{k}\times\frac{n!}{(k-1)!\,(n-(k-1))!}$$

$$= \frac{(n+1)!}{k!\,(n+1-k)!}$$

(2.)

$$S_n = \sum_{k=1}^{n}\frac{1}{k}\binom{n}{k-1} = \frac{1}{n+1}\sum_{k=1}^{n}\binom{n+1}{k} = \frac{1}{n+1}\left(\sum_{k=0}^{n+1}\binom{n+1}{k} - \binom{n+1}{0} - \binom{n+1}{n+1}\right)$$

$$= \frac{1}{n+1}\left(2^{n+1}-1-1\right) = \boxed{\frac{2^{n+1}-2}{n+1}}$$

Exercice 4

(1.) Par récurrence double : $\forall n\in\mathbb{N}$, $u_n > 0$.

$n=0$: $u_0 > 0$ ✓

$n=1$: $u_1 > 0$

récurrence initialisée

Supposons que $u_n > 0$ et $u_{n+1} > 0$ à un certain rang $n \in \mathbb{N}$.

Montrons que : $u_{n+2} > 0$.

$u_{n+2} = u_n^3 \times u_{n+1}^2$ or par (HR) : $\begin{cases} u_n > 0 \text{ donc } u_n^3 > 0 \\ u_{n+1} > 0 \text{ donc } u_{n+1}^2 > 0 \end{cases}$

donc $u_{n+2} > 0$ ✓

récurrence achevée.

(2.) (a) $\forall n \in \mathbb{N},\ u_n > 0$ donc (t_n) bien définie.

(b) $\forall n \in \mathbb{N}$,

$t_{n+2} = \ln(u_{n+2}) = \ln(u_n^3 \times u_{n+1}^2) = \ln(u_{n+1}^2) + \ln(u_n^3)$

$= 2\ln(u_{n+1}) + 3\ln(u_n)$

$t_{n+2} = 2t_{n+1} + 3t_n$ ✓

(c) Equation caractéristique : $x^2 = 2x + 3 \iff x^2 - 2x - 3 = 0$.

$\iff x = -1$ ou $x = 3$.

donc $\exists \lambda, \mu \in \mathbb{R} / \forall n \in \mathbb{N},\ t_n = \lambda(-1)^n + \mu 3^n$, avec :

$$\begin{cases} \lambda + \mu = \ln(u_0). \\ -\lambda + 3\mu = \ln(u_1) \end{cases} \underset{L_2 \leftarrow L_2 + L_1}{\iff} \begin{cases} \lambda + \mu = \ln(u_0) \\ 4\mu = \ln(u_1) + \ln(u_0) \end{cases}$$

$$\iff \begin{cases} \lambda = \ln(u_0) - \lambda = \ln(u_0) - \dfrac{\ln(u_0) + \ln(u_1)}{4} = \dfrac{3\ln(u_0) - \ln(u_1)}{4} \\ \mu = \dfrac{\ln(u_0) + \ln(u_1)}{4} = \dfrac{\ln(u_1 u_0)}{4}. \end{cases}$$

Donc : $\boxed{\forall n \in \mathbb{N},\ t_n = \dfrac{3\ln(u_0) - \ln(u_1)}{4}(-1)^n + \dfrac{\ln(u_0 u_1)}{4} 3^n}$

$\forall n \in \mathbb{N}$, $\quad u_n = e^{t_n} = e^{(-1)^n\left(\frac{3\ln u_0 - \ln u_1}{4}\right)} \, e^{\frac{3^n}{4}\ln\left(u_1 u_0\right)}$

$$\boxed{u_n = \left(\frac{u_0^3}{u_1}\right)^{\frac{(-1)^n}{4}} \times \left(u_1 u_0\right)^{\frac{3^n}{4}}}$$

Exercice 5

(1) (a) $\boxed{f'(x) = -\dfrac{2x + e^{-x}}{(x^2 - e^{-x})^2}}$

(b) $g'(x) = \dfrac{1}{2\sqrt{\ln(2x+1)}} \times \dfrac{2}{2x+1}$

$= \boxed{\dfrac{1}{(2x+1)\sqrt{\ln(2x+1)}}}$

(2.) (a) Ensemble de définition: $|x^2 - x + 2| \geq 0$ donc $\boxed{D_f = \mathbb{R}}$

(b) Domaine de dérivation: $x^2 - x - 2 \neq 0$ et $|x^2 - x - 2| \neq 0$

$\iff x^2 - x - 2 \neq 0.$

$\iff x \neq -1$ et $x \neq 2$

donc $\boxed{D_{f'} = \mathbb{R} \setminus \{-1, 2\}}$

(c)

• si $x \in]-\infty, -1[\cup]2, +\infty[$: $x^2 - x - 2 > 0$

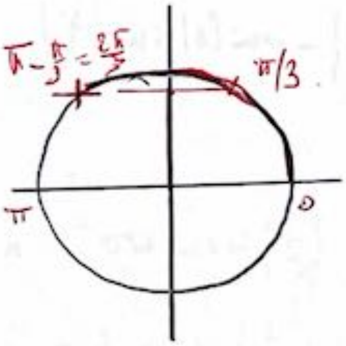
donc $f(x) = \sqrt{x^2 - x - 2}$ donc $\boxed{f'(x) = \dfrac{2x - 1}{2\sqrt{x^2 - x - 2}}}$

• si $x \in]-1, 2[$: $x^2 - x - 2 < 0$ donc $f(x) = \sqrt{-x^2+x+2}$

donc $\boxed{f'(x) = \dfrac{-2x+1}{2\sqrt{-x^2+x+2}}}$

Exercice 6. /21

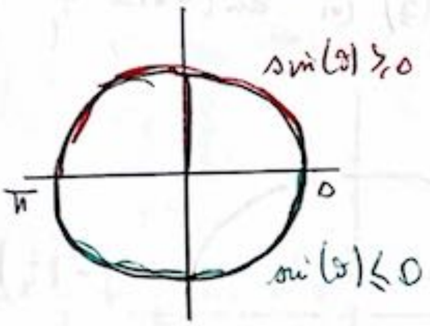
1. (a) /2



$0 \leq x < \frac{2\pi}{3}$ donc $\boxed{\sin(x) \geq 0}$

(b) /4

$0 \leq x < \frac{2\pi}{3}$ donc $0 \leq 3x < 2\pi$



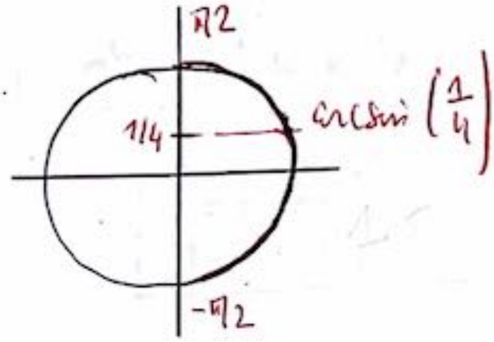
. $\sin(3x) \geq 0 \iff 0 < 3x < \pi \iff 0 < x < \frac{\pi}{3}$.

. $\sin(3x) = 0 \iff 3x = 0$ ou $3x = \pi$

$\iff x = 0$ ou $x = \frac{\pi}{3}$.

x	0		$\pi/3$		$2\pi/3$.
$\sin(3x)$	0	$+$	0	$-$	

(c)



donc $\arcsin\left(\frac{1}{4}\right) \in [0$

2. $\cos(a+b) = \cos a \cos b - \sin a \sin b$.

donc $\cos\left(\theta + \frac{\pi}{2}\right) = \cos\theta \underbrace{\cos\left(\frac{\pi}{2}\right)}_{=0} - \sin(\theta) \underbrace{\sin\left(\frac{\pi}{2}\right)}_{1} = -\sin(\theta)$ V

3.

$(3) \iff \left| -\sin(3x) \right| = \frac{1}{4} \iff \left| \sin(3x) \right| = \frac{1}{4}$.

* si $0 \le x \le \frac{\pi}{3}$: $\sin(3x) \geq 0$ donc $(3) \iff \sin(3x) = \frac{1}{4}$.

$0 \le 3x \le \pi$:

$(3) \iff 3x = \arcsin\left(\frac{1}{4}\right)$ ou $3x = \pi - \arcsin\left(\frac{1}{4}\right)$

$\iff x = \frac{1}{3}\arcsin\left(\frac{1}{4}\right)$ ou $x = \frac{\pi}{3} - \frac{1}{3}\arcsin\left(\frac{1}{4}\right)$

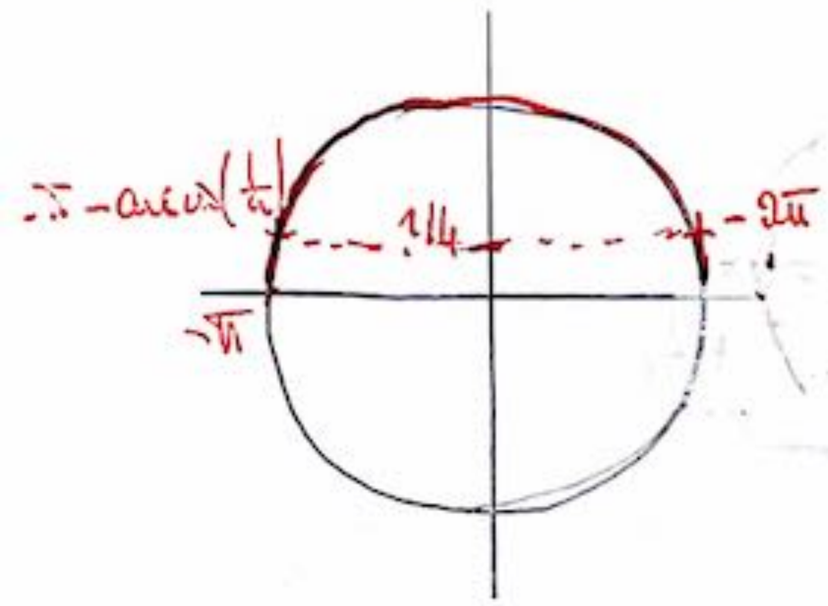
or $\arcsin\left(\frac{1}{4}\right) \in [0, \frac{\pi}{2}[$ donc $\frac{1}{3}\arcsin\left(\frac{1}{4}\right) \in [0, \frac{2\pi}{3}[$ V

et $-\frac{\pi}{6} < -\frac{1}{3}\arcsin\left(\frac{1}{4}\right) \le 0$ donc $0 \le \frac{\pi}{6} < \frac{\pi}{3} - \frac{1}{3}\arcsin\left(\frac{1}{4}\right) \le \frac{\pi}{3} < \frac{2\pi}{3}$

* si $\frac{\pi}{3} < x < \frac{2\pi}{3}$: $\sin(3x) < 0$ donc (3) $\Leftrightarrow -\sin(3x) = \frac{1}{4}$

$\Leftrightarrow \sin(-3x) = \frac{1}{4}$

$-2\pi < -3x < -\pi$



(3) $\Leftrightarrow \quad 3x = -\pi - \arcsin\left(\frac{1}{4}\right)$ ou $3x = -2\pi + \arcsin\left(\frac{1}{4}\right)$

$\Leftrightarrow \quad x = -\frac{\pi}{3} - \frac{1}{3}\arcsin\left(\frac{1}{4}\right)$ ou $x = -\frac{2\pi}{3} + \frac{1}{3}\arcsin\left(\frac{1}{4}\right)$

avec:

1/12 · $0 \leq \arcsin\left(\frac{1}{4}\right) < \frac{\pi}{2}$ donc $-\frac{\pi}{6} < -\frac{1}{3}\arcsin\left(\frac{1}{4}\right) \leq 0$

donc $-\frac{\pi}{3} - \frac{1}{3}\arcsin\left(\frac{1}{4}\right) < 0$ X

· $-\frac{2\pi}{3} \leq -\frac{2\pi}{3} + \frac{1}{3}\arcsin\left(\frac{1}{4}\right) < \underbrace{-\frac{2\pi}{3} + \frac{\pi}{6}}_{-\frac{3\pi}{6} = -\frac{\pi}{2}} < 0$ X

Conclusion :

$$\boxed{S = \left\{ \frac{1}{3}\arcsin\left(\frac{1}{4}\right) ; \frac{\pi}{3} - \frac{1}{3}\arcsin\left(\frac{1}{4}\right) \right\}}$$