DS3 - Mathématiques -
Corrigé
12/11/2025

## Exercice 2

(1.) $$f'(x) = \frac{1}{2x - \frac{3}{x}} \times \left(2 + \frac{3}{x^2}\right) = \frac{x}{2x^2-3} \times \left(\frac{3+2x^2}{x^2}\right)$$

$$\boxed{f'(x) = \frac{3+2x^2}{x(2x^2-3)}}$$

(2.) $$g'(x) = \frac{1}{(x^2-1)^2} \times \left(\frac{\cancel{2}e^{2x}}{\cancel{2}\sqrt{e^{2x}}}(x^2-1) - \sqrt{e^{2x}} \times 2x\right)$$

$$\boxed{g'(x) = \frac{\sqrt{e^{2x}}}{(x^2-1)^2}\left(x^2-1-2x\right)}$$

(3.) $$h'(x) = e^{\frac{1}{\ln(x)}} \times \left(-\frac{\frac{1}{x}}{(\ln x)^2}\right)$$

$$\boxed{h'(x) = -\frac{1}{x(\ln(x))^2}e^{\frac{1}{\ln(x)}}}$$

Exercice 3

1. $\forall n \in \mathbb{N}$,

$$S_n = \frac{1}{2} \times \frac{1}{3^{2n+1}} \sum_{k=0}^{n} \binom{n}{k} \frac{2^k}{3^k} = \frac{1}{2\times 3\times 9^n} \sum_{k=0}^{n} \binom{n}{k} \left(\frac{2}{3}\right)^k 1^{n-k}$$

$$= \frac{1}{6\times 9^n} \left(\frac{2}{3}+1\right)^n \qquad \text{donc} \quad \boxed{S_n = \frac{1}{6\times 9^n} \times \left(\frac{5}{3}\right)^n}$$

2. $\forall n \geq 2$,

$$R_n \underset{k'=k+2}{=} \sum_{k=2}^{n+2} \binom{n}{k} \left(\frac{-1}{3}\right)^{k-2} = \left(-\frac{1}{3}\right)^2 \sum_{k=2}^{n+2} \binom{n}{k} \left(\frac{-1}{3}\right)^k$$

$$= \frac{1}{9} \left( \sum_{k=2}^{n} \binom{n}{k} \left(-\frac{1}{3}\right)^k + 0 + 0 \right) \qquad \text{car } \binom{n}{n+1} = 0 \text{ et } \binom{n}{n+2} = 0$$

$$= \frac{1}{9} \left( \sum_{k=0}^{n} \binom{n}{k} \left(-\frac{1}{3}\right)^k - \binom{n}{0} \left(-\frac{1}{3}\right)^0 - \binom{n}{1} \left(-\frac{1}{3}\right)^1 \right)$$

$$= \frac{1}{9} \left( \left(-\frac{1}{3}+1\right)^n - 1 - n\left(-\frac{1}{3}\right) \right)$$

$$\boxed{R_n = \frac{1}{9} \left( \left(\frac{2}{3}\right)^n - 1 + \frac{n}{3} \right)}$$

3. $\forall n \in \mathbb{N}$,

$$T_n = \sum_{k=1}^{n+1} \binom{n+1}{k} 2^{n-k} (-1)^{k-1} + 0 \qquad \text{car } \binom{n+1}{n+2} = 0$$

$$= (-1)^{-1} \times \frac{1}{2} \sum_{k=1}^{n+1} \binom{n+1}{k} 2^{n+1-k} (-1)^k$$

$$= -\frac{1}{2} \left( \sum_{k=0}^{n+1} \binom{n+1}{k} 2^{n+1-k} (-1)^k - \binom{n+1}{0} 2^{n+1-0} (-1)^0 \right)$$

$$T_n = -\frac{1}{2}\left( (-1+2)^{n+1} - 2^{n+1} \right) \quad \text{donc } T_n = -\frac{1}{2}\left(1 - 2^{n+1}\right)$$

$$\boxed{T_n = 2^n - \frac{1}{2}}$$

(4.) $\forall n \geq 2,$

par symétrie, $W_n = \sum_{k=0}^{n} k(n-k)\binom{n}{n-k} = \sum_{k=0}^{n-1} k(n-k)\binom{n}{n-k} + 0$

par la formule du chef : $\binom{n}{n-k} = \frac{n}{n-k}\binom{n-1}{n-k-1}$ , $(k \neq n)$

$$W_n = \sum_{k=0}^{n-1} k\, n \binom{n-1}{(n-1)-k} = n \sum_{k=0}^{n-1} k \binom{n-1}{(n-1)-k}$$

par symétrie, $W_n = n \sum_{k=0}^{n-1} k \binom{n-1}{k} = n \sum_{k=1}^{n-1} k \binom{n-1}{k} + 0$

par la formule du chef : $\binom{n-1}{k} = \frac{n-1}{k}\binom{n-2}{k-1}$ , $(k \neq 0)$

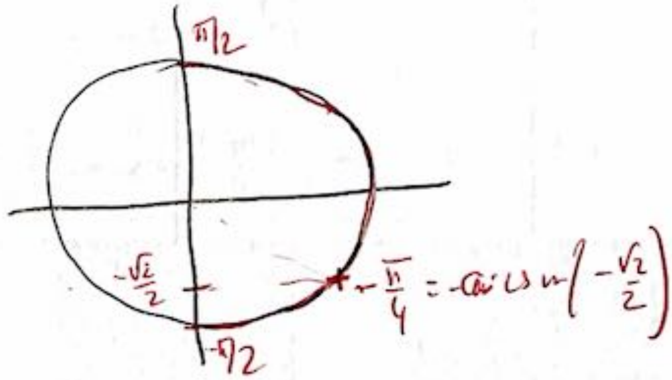
$$W_n = n \sum_{k=1}^{n-1} (n-1)\binom{n-2}{k-1} = n(n-1) \sum_{k=1}^{n-1} \binom{n-2}{k-1}$$

$$\underset{k'=k-1}{=} n(n-1) \sum_{k=0}^{n-2} \binom{n-2}{k}$$

donc $\boxed{W_n = n(n-1)\, 2^{n-2}}$

Exercice 4

1.



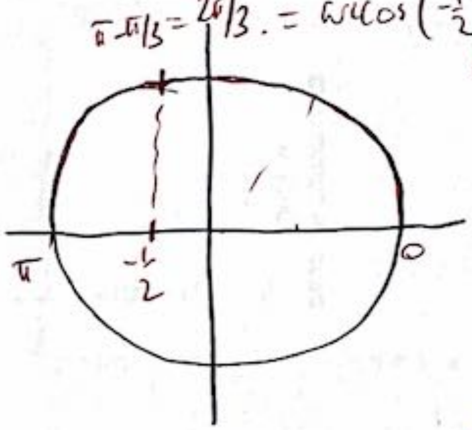

sur $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, l'équation $\arcsin(x) = -\frac{\sqrt{2}}{2}$ a une unique solution :

$$x = \arcsin\left(-\frac{\sqrt{2}}{2}\right)$$

or $\sin\left(-\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$ et $-\frac{\pi}{4} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

donc par unicité, $\boxed{\arcsin\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}}$

2.




sur $[0, \pi]$, l'équation $\cos(x) = -\frac{1}{2}$ admet une unique solution :

$$x = \arccos\left(-\frac{1}{2}\right)$$

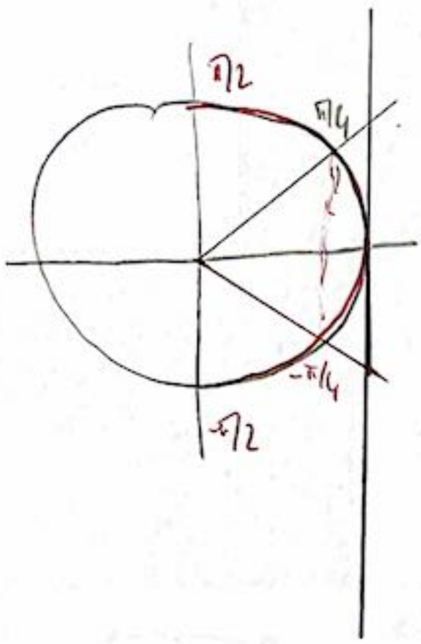
or $\cos\left(\pi - \frac{\pi}{3}\right) = -\cos\left(\frac{\pi}{3}\right) = -\frac{1}{2}$ et $\pi - \frac{\pi}{3} = \frac{2\pi}{3} \in [0, \pi]$.

donc par unicité : $\boxed{\arccos\left(-\frac{1}{2}\right) = \frac{2\pi}{3}}$

(3.) sur $]-\frac{\pi}{2}, \frac{\pi}{2}[$, l'équation $\tan(x) = -1$ admet une unique solution : $x = \arctan(-1)$

or $\tan\left(-\frac{\pi}{4}\right) = -\tan\left(\frac{\pi}{4}\right) = -1$ et $-\frac{\pi}{4} \in ]-\frac{\pi}{2}, \frac{\pi}{2}[$

donc par unicité : $\boxed{\arctan(-1) = -\frac{\pi}{4}}$




Exercice 5

(1.) On rappelle que $\sin(a+b) = \sin a \cos b + \cos a \sin b \quad \forall a,b \in \mathbb{R}$

donc pour $a = b = \theta \in \mathbb{R}$,

$$\sin(2\theta) = \sin\theta\cos\theta + \cos\theta\sin(\theta) = 2\sin\theta\cos\theta \quad ✓$$

(2.) Par récurrence sur $n \in \mathbb{N}^*$ : $\prod_{k=1}^{n} \cos\left(\frac{a}{2^k}\right) = \frac{1}{2^n} \frac{\sin(a)}{\sin\left(\frac{a}{2^n}\right)}$

$\underline{\text{si } n = 1}$ $\prod_{k=1}^{1} \cos\left(\frac{a}{2^k}\right) = \cos\left(\frac{a}{2}\right)$

et $\frac{1}{2^n} \frac{\sin(a)}{\sin\left(\frac{a}{2^1}\right)} = \frac{\sin(a)}{2\sin\left(\frac{a}{2}\right)} = \frac{\cancel{2}\sin\cancel{\left(\frac{a}{2}\right)}\cos\left(\frac{a}{2}\right)}{\cancel{2}\sin\cancel{\left(\frac{a}{2}\right)}}$ (cf 1.)

$= \cos\left(\frac{a}{2}\right)$ → récurrence initialisée

$\underline{\bullet\ n \geq 1}$ : Supposons que $\prod_{k=1}^{n} \cos\left(\frac{a}{2^k}\right) = \frac{\sin(a)}{2^n \sin\left(\frac{a}{2^n}\right)}$ à un certain rang $n$.

Montrons que $\prod_{k=1}^{n+1} \cos\left(\frac{a}{2^k}\right) = \frac{\sin(a)}{2^{n+1} \sin\left(\frac{a}{2^{n+1}}\right)}$.

$$\prod_{k=1}^{n+1} \cos\left(\frac{a}{2^k}\right) = \prod_{k=1}^{n} \cos\left(\frac{a}{2^k}\right) \times \cos\left(\frac{a}{2^{n+1}}\right)$$

$$\underset{(HR)}{=} \frac{\sin(a)}{2^n \sin\left(\frac{a}{2^n}\right)} \times \cos\left(\frac{a}{2^{n+1}}\right)$$

or $\sin\left(\frac{a}{2^n}\right) = \sin\left(2 \times \frac{a}{2^{n+1}}\right) \underset{1.}{=} 2 \sin\left(\frac{a}{2^{n+1}}\right) \cos\left(\frac{a}{2^{n+1}}\right)$

donc $\prod_{k=1}^{n+1} \cos\left(\frac{a}{2^k}\right) = \frac{\sin(a)}{2^n} \times \frac{\cancel{\cos\left(\frac{a}{2^{n+1}}\right)}}{2 \sin\left(\frac{a}{2^{n+1}}\right) \cancel{\cos\left(\frac{a}{2^{n+1}}\right)}}$

$$= \frac{\sin(a)}{2^{n+1} \sin\left(\frac{a}{2^{n+1}}\right)}$$ → récurrence achevée.

## $\boxed{\text{Exercice 6.}}$

1. On résout pour $x \neq \frac{\pi}{2}$ et $x \neq \frac{3\pi}{2}$

$(1) \Leftrightarrow \frac{\sin(x)}{\cos x} = 2\sin(x) \Leftrightarrow \sin(x)\left(\frac{1}{\cos x} - 2\right) = 0$

$\Leftrightarrow \sin x = 0$ ou $\frac{1-2\cos x}{\cos x} = 0$.

$\Leftrightarrow x = 0$ ou $x = \pi$ ou $1 - 2\cos x = 0$.

$1-2\cos x = 0 \iff \cos x = \frac{1}{2} \iff x = \frac{\pi}{3}$ ou $x = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}.$

$0, \pi, \frac{\pi}{3}$ et $\frac{5\pi}{3}$ appartiennent à l'ensemble de résolution

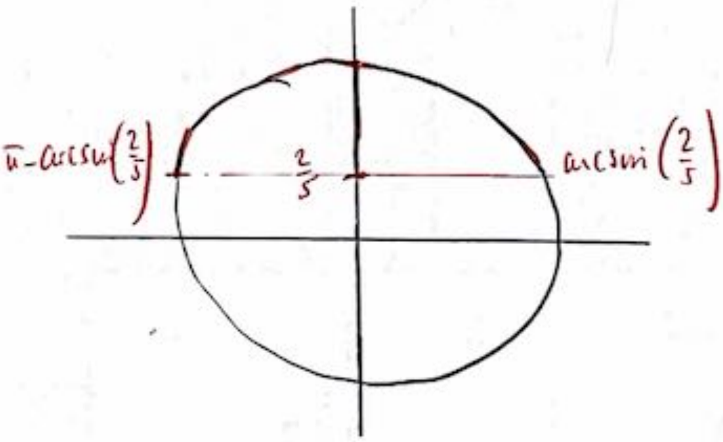
donc $\boxed{S = \left\{0, \pi, \frac{\pi}{3}, \frac{5\pi}{3}\right\}}$

(2.) $\forall x \in [0, 2\pi[,\quad (2) \iff \frac{\sin(2x)}{2} > \frac{1}{5} \iff \sin(2x) > \frac{2}{5}$

soit $X = 2x$

$0 \leq x < 2\pi$

$\iff 0 \leq X < 4\pi$ (2 tours)




donc $(2) \iff \left|\begin{array}{l} \arcsin\left(\frac{2}{5}\right) < X < \pi - \arcsin\left(\frac{2}{5}\right) \\ \text{ou} \\ 2\pi + \arcsin\left(\frac{2}{5}\right) < X < 2\pi + \pi - \arcsin\left(\frac{2}{5}\right) \end{array}\right.$

$\iff \left|\begin{array}{l} \frac{1}{2}\arcsin\left(\frac{2}{5}\right) < x < \frac{\pi}{2} - \frac{1}{2}\arcsin\left(\frac{2}{5}\right) \\ \text{ou} \\ \pi + \frac{1}{2}\arcsin\left(\frac{2}{5}\right) < x < \pi + \frac{\pi}{2} - \frac{1}{2}\arcsin\left(\frac{2}{5}\right) \end{array}\right. \qquad \left(x = \frac{X}{2}\right)$

$$\boxed{S = \left]\frac{\arcsin\left(\frac{2}{5}\right)}{2}, \frac{\pi}{2} - \frac{\arcsin\left(\frac{2}{5}\right)}{2}\right[ \cup \left]\pi + \frac{\arcsin\left(\frac{2}{5}\right)}{2}, \frac{3\pi}{2} - \frac{\arcsin\left(\frac{2}{5}\right)}{2}\right[}$$

(3.) on pose $X = \cos(x) \in [-1, 1]$.

$(3) \Leftrightarrow 2X^2 + 9X + 4 = 0$.

$$\Delta = 81 - 2 \times 4 \times 4 = 81 - 32 = 49 = 7^2$$

$(3) \Leftrightarrow X = \frac{-9-7}{4} = -\frac{16}{4} = -4 \notin [-1, 1]$ ou $X = \frac{-9+7}{4} = -\frac{2}{4} = -\frac{1}{2} \in [-1, 1]$

$\Leftrightarrow \cos x = -\frac{1}{2} = \cos\left(\pi + \frac{\pi}{3}\right) = \cos\left(\frac{4\pi}{3}\right)$

$$\Leftrightarrow \boxed{\begin{array}{|l} x = \frac{4\pi}{3} + 2k\pi,\ k \in \mathbb{Z} \\ \text{ou } x = -\frac{4\pi}{3} + 2\ell\pi,\ \ell \in \mathbb{Z} \end{array}}$$