1 Exercices

1.1 Systèmes de Cramer

Ces systèmes apparaissent lorsque l'on démontre qu'une famille de vecteurs est libre (exercice 1) ou lorsque l'on résout un problème de Cauchy (exercice 2).

Exercice 1 : Résoudre les systèmes linéaires homogènes suivants.

1. $\begin{cases} x + y + 2z = 0 \\ x + 2y + z = 0 \\ 2x + y + z = 0 \end{cases}$

3. $\begin{cases} x + y + 2z = 0 \\ x - y - z = 0 \\ x + z = 0 \end{cases}$

5. $\begin{cases} y + z = 0 \\ x - y + 2z = 0 \\ 2x + 3y - z = 0 \end{cases}$

2. $\begin{cases} x + 2z = 0 \\ - y + z = 0 \\ x - 2y = 0 \end{cases}$

4. $\begin{cases} x + y - z = 0 \\ x - y = 0 \\ x + 4y + z = 0 \end{cases}$

6. $\begin{cases} - 3y + 2z = 0 \\ 4x - z = 0 \\ -3x - 3z = 0 \end{cases}$

Exercice 2 : Résoudre les systèmes linéaires avec second membre suivants.

1. $\begin{cases} x + y + 2z = 3 \\ x + 2y + z = 1 \\ 2x + y + z = 0 \end{cases}$

2. $\begin{cases} x + 2z = 1 \\ - y + z = 2 \\ x - 2y = 1 \end{cases}$

3. $\begin{cases} x + y + 2z = 5 \\ x - y - z = 1 \\ x + z = 3 \end{cases}$

1.2 Systèmes se résolvant à l'aide de deux variables auxiliaires

Ces systèmes apparaissent lorsque l'on calcule un sous-espace propre de dimension 2.

Exercice 3 : Résoudre les systèmes linéaires suivants.

1. $\begin{cases} 3x - 2y + z = 0 \\ 6x - 4y + 2z = 0 \\ -3x + 2y - z = 0 \end{cases}$

3. $\begin{cases} -2x + y + z = 0 \\ -4x + 2y + 2z = 0 \\ 4x - 2y - 2z = 0 \end{cases}$

5. $\begin{cases} 0 = 0 \\ -z = 0 \\ -4z = 0 \end{cases}$

2. $\begin{cases} x + 4y - z = 0 \\ -2x - 8y + 2z = 0 \\ 3x + 12y - 3z = 0 \end{cases}$

4. $\begin{cases} -2x - 2y + 2z = 0 \\ -x - y + z = 0 \\ 4x + 4y - 4z = 0 \end{cases}$

6. $\begin{cases} -2x - y = 0 \\ -4x - 2y = 0 \\ 2x + y = 0 \end{cases}$

1.3 Systèmes se résolvant à l'aide d'une seule variable auxiliaire

Ces systèmes apparaissent lorsque l'on calcule un sous-espace propre de dimension 1.

Exercice 4 : Résoudre les systèmes linéaires suivants.

1. $\begin{cases} 2x + 2y + z = 0 \\ -x + 2y + 3z = 0 \\ x + 4y + 4z = 0 \end{cases}$

2. $\begin{cases} 3x + y - z = 0 \\ x - 2y + 2z = 0 \\ x + y - z = 0 \end{cases}$

3. $\begin{cases} 2x + 2y + z = 0 \\ -x + 2y + z = 0 \\ x + 4y + 2z = 0 \end{cases}$

4. $\begin{cases} 2x - y - z = 0 \\ -x + 2y - z = 0 \\ 3x - 3z = 0 \end{cases}$ 5. $\begin{cases} 3y + 5z = 0 \\ x + 4y + 2z = 0 \\ -x - y + 3z = 0 \end{cases}$ 6. $\begin{cases} -x + 3y - z = 0 \\ 2x - 7y + 2z = 0 \\ -x + 2y - z = 0 \end{cases}$

2 Réponses courtes

Réponses de l'exercice 1 :

1. $\begin{cases} x + y + 2z = 0 \\ x + 2y + z = 0 \\ 2x + y + z = 0 \end{cases} \iff x = y = z = 0$

2. $\begin{cases} x + 2z = 0 \\ - y + z = 0 \\ x - 2y = 0 \end{cases} \iff x = y = z = 0$

3. $\begin{cases} x + y + 2z = 0 \\ x - y - z = 0 \\ x + z = 0 \end{cases} \iff x = y = z = 0$

4. $\begin{cases} x + y - z = 0 \\ x - y = 0 \\ x + 4y + z = 0 \end{cases} \iff x = y = z = 0$

5. $\begin{cases} y + z = 0 \\ x - y + 2z = 0 \\ 2x + 3y - z = 0 \end{cases} \iff x = y = z = 0$

6. $\begin{cases} - 3y + 2z = 0 \\ 4x - z = 0 \\ -3x - 3z = 0 \end{cases} \iff x = y = z = 0$

Réponses de l'exercice 2 :

1. $\begin{cases} x + y + 2z = 3 \\ x + 2y + z = 1 \\ 2x + y + z = 0 \end{cases} \iff \begin{cases} x = -1 \\ y = 0 \\ z = 2 \end{cases}$

2. $\begin{cases} x + 2z = 1 \\ - y + z = 2 \\ x - 2y = 1 \end{cases} \iff \begin{cases} x = -1 \\ y = -1 \\ z = 1 \end{cases}$

3. $\begin{cases} x + y + 2z = 5 \\ x - y - z = 1 \\ x + z = 3 \end{cases} \iff \begin{cases} x = 3 \\ y = 2 \\ z = 0 \end{cases}$

Réponses de l'exercice 3 :

1. $\begin{cases} 3x - 2y + z = 0 \\ 6x - 4y + 2z = 0 \\ -3x + 2y - z = 0 \end{cases} \iff z = 2y - 3x$

2. $\begin{cases} x + 4y - z = 0 \\ -2x - 8y + 2z = 0 \\ 3x + 12y - 3z = 0 \end{cases} \iff z = x + 4y$

3. $\begin{cases} -2x + y + z = 0 \\ -4x + 2y + 2z = 0 \\ 4x - 2y - 2z = 0 \end{cases} \iff z = 2x - y$

4. $\begin{cases} -2x - 2y + 2z = 0 \\ -x - y + z = 0 \\ 4x + 4y - 4z = 0 \end{cases} \iff z = x + y$

5. $\begin{cases} 0 = 0 \\ -z = 0 \\ -4z = 0 \end{cases} \iff z = 0$

6. $\begin{cases} -2x - y = 0 \\ -4x - 2y = 0 \\ 2x + y = 0 \end{cases} \iff y = -2x$

Réponses de l'exercice 4 :

1. $\begin{cases} 2x + 2y + z = 0 \\ -x + 2y + 3z = 0 \\ x + 4y + 4z = 0 \end{cases} \iff \begin{cases} x = \frac{2}{3}z \\ y = -\frac{7}{6}z \end{cases}$

2. $\begin{cases} 3x + y - z = 0 \\ x - 2y + 2z = 0 \\ x + y - z = 0 \end{cases} \iff \begin{cases} x = 0 \\ y = z \end{cases}$

3. $\begin{cases} 2x + 2y + z = 0 \\ -x + 2y + z = 0 \\ x + 4y + 2z = 0 \end{cases} \iff \begin{cases} x = 0 \\ y = -\frac{1}{2}z \end{cases}$

5. $\begin{cases} 3y + 5z = 0 \\ x + 4y + 2z = 0 \\ -x - y + 3z = 0 \end{cases} \iff \begin{cases} x = \frac{14}{3}z \\ y = -\frac{5}{3}z \end{cases}$

4. $\begin{cases} 2x - y - z = 0 \\ -x + 2y - z = 0 \\ 3x - 3z = 0 \end{cases} \iff \begin{cases} x = z \\ y = z \end{cases}$

6. $\begin{cases} -x + 3y - z = 0 \\ 2x - 7y + 2z = 0 \\ -x + 2y - z = 0 \end{cases} \iff \begin{cases} x = -z \\ y = 0 \end{cases}$

3   Corrections détaillées

Correction détaillée de l'exercice 1 :

1.

$$\begin{cases} x + y + 2z = 0 \\ x + 2y + z = 0 \\ 2x + y + z = 0 \end{cases} \iff \begin{cases} x + y + 2z = 0 \\ y - z = 0 \\ -y - 3z = 0 \end{cases} \qquad \begin{array}{l} L_2 \leftarrow L_2 - L_1 \\ L_3 \leftarrow L_3 - 2L_1 \end{array}$$

$$\iff \begin{cases} x + y + 2z = 0 \\ y - z = 0 \\ - 4z = 0 \end{cases} \qquad L_3 \leftarrow L_3 + L_2$$

$$\iff \begin{cases} x = 0 \\ y = 0 \\ z = 0 \end{cases} \qquad \text{(par remontées successives)}$$

2.

$$\begin{cases} x + 2z = 0 \\ - y + z = 0 \\ x - 2y = 0 \end{cases} \iff \begin{cases} x + 2z = 0 \\ - y + z = 0 \\ - 2y - 2z = 0 \end{cases} \qquad L_3 \leftarrow L_3 - L_1$$

$$\iff \begin{cases} x + 2z = 0 \\ - y + z = 0 \\ - 4z = 0 \end{cases} \qquad L_3 \leftarrow L_3 - 2L_2$$

$$\iff \begin{cases} x = 0 \\ y = 0 \\ z = 0 \end{cases} \qquad \text{(par remontées successives)}$$

3.

$$\begin{cases} x + y + 2z = 0 \\ x - y - z = 0 \\ x + z = 0 \end{cases} \iff \begin{cases} x + y + 2z = 0 \\ -2y - 3z = 0 \\ -y - z = 0 \end{cases} \qquad \begin{array}{l} L_2 \leftarrow L_2 - L_1 \\ L_3 \leftarrow L_3 - L_1 \end{array}$$

$$\iff \begin{cases} x + y + 2z = 0 \\ -2y - 3z = 0 \\ z = 0 \end{cases} \qquad L_3 \leftarrow 2L_3 - L_2$$

$$\iff \begin{cases} x = 0 \\ y = 0 \\ z = 0 \end{cases} \qquad \text{(par remontées successives)}$$

4.

$$\begin{cases} x + y - z = 0 \\ x - y = 0 \\ x + 4y + z = 0 \end{cases} \iff \begin{cases} x + y - z = 0 \\ -2y + z = 0 \\ 3y + 2z = 0 \end{cases} \qquad \begin{array}{l} L_2 \leftarrow L_2 - L_1 \\ L_3 \leftarrow L_3 - L_1 \end{array}$$

$$\iff \begin{cases} x + y - z = 0 \\ -2y + z = 0 \\ 7z = 0 \end{cases} \qquad L_3 \leftarrow 2L_3 + 3L_2$$

$$\iff \begin{cases} x = 0 \\ y = 0 \\ z = 0 \end{cases} \qquad \text{(par remontées successives)}$$

5.

$$\begin{cases} y + z = 0 \\ x - y + 2z = 0 \\ 2x + 3y - z = 0 \end{cases} \iff \begin{cases} x - y + 2z = 0 \\ y + z = 0 \\ 2x + 3y - z = 0 \end{cases} \qquad L_1 \leftrightarrow L_2$$

$$\iff \begin{cases} x - y + 2z = 0 \\ y + z = 0 \\ 5y - 5z = 0 \end{cases} \qquad L_3 \leftarrow L_3 - 2L_1$$

$$\iff \begin{cases} x - y + 2z = 0 \\ y + z = 0 \\ - 10z = 0 \end{cases} \qquad L_3 \leftarrow L_3 - 5L_2$$

$$\iff \begin{cases} x = 0 \\ y = 0 \\ z = 0 \end{cases} \qquad \text{(par remontées successives)}$$

6.

$$\begin{cases} - 3y + 2z = 0 \\ 4x - z = 0 \\ -3x - 3z = 0 \end{cases} \iff \begin{cases} 4x - z = 0 \\ - 3y + 2z = 0 \\ -3x - 3z = 0 \end{cases} \qquad L_1 \leftrightarrow L_2$$

$$\iff \begin{cases} 4x - z = 0 \\ - 3y + 2z = 0 \\ - 15z = 0 \end{cases} \qquad L_3 \leftarrow 4L_3 + 3L_1$$

$$\iff \begin{cases} x = 0 \\ y = 0 \\ z = 0 \end{cases} \qquad \text{(par remontées successives)}$$

Correction détaillée de l'exercice 2 :

1.

$$\begin{cases} x + y + 2z = 3 \\ x + 2y + z = 1 \\ 2x + y + z = 0 \end{cases} \iff \begin{cases} x + y + 2z = 3 \\ y - z = -2 \\ - y - 3z = -6 \end{cases} \qquad \begin{array}{l} L_2 \leftarrow L_2 - L_1 \\ L_3 \leftarrow L_3 - 2L_1 \end{array}$$

$$\iff \begin{cases} x + y + 2z = 3 \\ y - z = -2 \\ - 4z = -8 \end{cases} \qquad L_3 \leftarrow L_3 + L_2$$

$$\iff \begin{cases} x = -1 \\ y = 0 \\ z = 2 \end{cases} \qquad \text{(par remontées successives)}$$

2.

$$\begin{cases} x + 2z = 1 \\ - y + z = 2 \\ x - 2y = 1 \end{cases} \iff \begin{cases} x + 2z = 1 \\ - y + z = 2 \\ - 2y - 2z = 0 \end{cases} \qquad L_3 \leftarrow L_3 - L_1$$

$$\iff \begin{cases} x + 2z = 1 \\ - y + z = 2 \\ - 4z = -4 \end{cases} \qquad L_3 \leftarrow L_3 - 2L_2$$

$$\iff \begin{cases} x = -1 \\ y = -1 \\ z = 1 \end{cases} \qquad \text{(par remontées successives)}$$

3.

$$\begin{cases} x + y + 2z = 5 \\ x - y - z = 1 \\ x + z = 3 \end{cases} \iff \begin{cases} x + y + 2z = 5 \\ -2y - 3z = -4 \\ -y - z = -2 \end{cases} \qquad \begin{array}{l} L_2 \leftarrow L_2 - L_1 \\ L_3 \leftarrow L_3 - L_1 \end{array}$$

$$\iff \begin{cases} x + y + 2z = 5 \\ -2y - 3z = -4 \\ z = 0 \end{cases} \qquad L_3 \leftarrow 2L_3 - L_2$$

$$\iff \begin{cases} x = 3 \\ y = 2 \\ z = 0 \end{cases} \qquad \text{(par remontées successives)}$$

Correction détaillée de l'exercice 3 :

1.

$$\begin{cases} 3x - 2y + z = 0 \\ 6x - 4y + 2z = 0 \\ -3x + 2y - z = 0 \end{cases} \iff 3x - 2y + z = 0 \qquad \begin{array}{l} \text{car } L_2 = 2L_1 \\ \text{et } L_3 = -L_1 \end{array}$$

$$\iff z = 2y - 3x$$

2.

$$\begin{cases} x + 4y - z = 0 \\ -2x - 8y + 2z = 0 \\ 3x + 12y - 3z = 0 \end{cases} \iff x + 4y - z = 0 \qquad \begin{array}{l} \text{car } L_2 = -2L_1 \\ \text{et } L_3 = 3L_1 \end{array}$$

$$\iff z = x + 4y$$

3.

$$\begin{cases} -2x + y + z = 0 \\ -4x + 2y + 2z = 0 \\ 4x - 2y - 2z = 0 \end{cases} \iff -2x + y + z = 0 \qquad \begin{array}{l} \text{car } L_2 = 2L_1 \\ \text{et } L_3 = -2L_1 \end{array}$$

$$\iff z = 2x - y$$

4.

$$\begin{cases} -2x - 2y + 2z = 0 \\ -x - y + z = 0 \\ 4x + 4y - 4z = 0 \end{cases} \iff x + y - z = 0 \qquad \begin{array}{l} \text{car } L_1 = 2L_2 \\ \text{et } L_3 = -4L_2 \end{array}$$

$$\iff z = x + y$$

5.

$$\begin{cases} 0 = 0 \\ -z = 0 \\ -4z = 0 \end{cases} \iff z = 0$$

6.

$$\begin{cases} -2x - y = 0 \\ -4x - 2y = 0 \\ 2x + y = 0 \end{cases} \iff 2x + y = 0 \qquad \begin{array}{l} \text{car } L_1 = -L_3 \\ \text{et } L_2 = -2L_3 \end{array}$$

$$\iff y = -2x$$

Correction détaillée de l'exercice 4 :

1.

$$\begin{cases} 2x + 2y + z = 0 \\ -x + 2y + 3z = 0 \\ x + 4y + 4z = 0 \end{cases} \iff \begin{cases} 2x + 2y + z = 0 \\ 6y + 7z = 0 \\ 6y + 7z = 0 \end{cases} \qquad \begin{array}{l} L_2 \leftarrow 2L_2 + L_1 \\ L_3 \leftarrow 2L_3 - L_1 \end{array}$$

$$\iff \begin{cases} 2x + 2y + z = 0 \\ 6y + 7z = 0 \\ 0 = 0 \end{cases} \qquad L_3 \leftarrow L_3 - L_2$$

$$\iff \begin{cases} 2x + 2y = -z \\ 6y = -7z \end{cases}$$

$$\iff \begin{cases} x = \frac{2}{3}z \\ y = -\frac{7}{6}z \end{cases} \qquad \text{(par remontées successives)}$$

2.

$$\begin{cases} 3x + y - z = 0 \\ x - 2y + 2z = 0 \\ x + y - z = 0 \end{cases} \iff \begin{cases} 3x + y - z = 0 \\ -7y + 7z = 0 \\ 2y - 2z = 0 \end{cases} \qquad \begin{array}{l} L_2 \leftarrow 3L_2 - L_1 \\ L_3 \leftarrow 3L_3 - L_1 \end{array}$$

$$\iff \begin{cases} 3x + y - z = 0 \\ -7y + 7z = 0 \\ 0 = 0 \end{cases} \qquad L_3 \leftarrow 7L_3 + 2L_2$$

$$\iff \begin{cases} 3x + y = z \\ y = z \end{cases}$$

$$\iff \begin{cases} x = 0 \\ y = z \end{cases} \qquad \text{(par remontées successives)}$$

3.

$$\begin{cases} 2x + 2y + z = 0 \\ -x + 2y + z = 0 \\ x + 4y + 2z = 0 \end{cases} \iff \begin{cases} 2x + 2y + z = 0 \\ 6y + 3z = 0 \\ 6y + 3z = 0 \end{cases} \qquad \begin{array}{l} L_2 \leftarrow 2L_2 + L_1 \\ L_3 \leftarrow 2L_3 - L_1 \end{array}$$

$$\iff \begin{cases} 2x + 2y + z = 0 \\ 6y + 3z = 0 \\ 0 = 0 \end{cases} \qquad L_3 \leftarrow L_3 - L_2$$

$$\iff \begin{cases} 2x + 2y = -z \\ 2y = -z \end{cases}$$

$$\iff \begin{cases} x = 0 \\ y = -\frac{1}{2}z \end{cases} \qquad \text{(par remontées successives)}$$

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4.

$$\begin{cases} 2x - y - z = 0 \\ -x + 2y - z = 0 \\ 3x - 3z = 0 \end{cases} \iff \begin{cases} 2x - y - z = 0 \\ 3y - 3z = 0 \\ 3y - 3z = 0 \end{cases} \qquad \begin{array}{l} L_2 \leftarrow 2L_2 + L_1 \\ L_3 \leftarrow 2L_3 - 3L_1 \end{array}$$

$$\iff \begin{cases} 2x - y - z = 0 \\ 3y - 3z = 0 \\ 0 = 0 \end{cases} \qquad L_3 \leftarrow L_3 - L_2$$

$$\iff \begin{cases} 2x - y = z \\ y = z \end{cases}$$

$$\iff \begin{cases} x = z \\ y = z \end{cases} \qquad \text{(par remontées successives)}$$

5.

$$\begin{cases} 3y + 5z = 0 \\ x + 4y + 2z = 0 \\ -x - y + 3z = 0 \end{cases} \iff \begin{cases} x + 4y + 2z = 0 \\ 3y + 5z = 0 \\ -x - y + 3z = 0 \end{cases} \qquad L_1 \leftrightarrow L_2$$

$$\iff \begin{cases} x + 4y + 2z = 0 \\ 3y + 5z = 0 \\ 3y + 5z = 0 \end{cases} \qquad L_3 \leftarrow L_3 + L_1$$

$$\iff \begin{cases} x + 4y + 2z = 0 \\ 3y + 5z = 0 \\ 0 = 0 \end{cases} \qquad L_3 \leftarrow L_3 - L_2$$

$$\iff \begin{cases} x + 4y = -2z \\ 3y = -5z \end{cases}$$

$$\iff \begin{cases} x = \frac{14}{3}z \\ y = -\frac{5}{3}z \end{cases} \qquad \text{(par remontées successives)}$$

6.

$$\begin{cases} -x + 3y - z = 0 \\ 2x - 7y + 2z = 0 \\ -x + 2y - z = 0 \end{cases} \iff \begin{cases} -x + 3y - z = 0 \\ - y = 0 \\ - y = 0 \end{cases} \qquad \begin{array}{l} L_2 \leftarrow L_2 + 2L_1 \\ L_3 \leftarrow L_3 - L_1 \end{array}$$

$$\iff \begin{cases} -x + 3y - z = 0 \\ - y = 0 \\ 0 = 0 \end{cases} \qquad L_3 \leftarrow L_3 - L_2$$

$$\iff \begin{cases} x = -z \\ y = 0 \end{cases} \qquad \text{(par remontées successives)}$$