Exercice 1 : Calculer le rang des matrices suivantes :

1. $\begin{pmatrix} 1 & 2 & 1 \\ 3 & 3 & 3 \\ -1 & 0 & 5 \end{pmatrix}$
2. $\begin{pmatrix} 1 & 4 & -3 \\ -4 & -1 & -2 \\ -1 & -2 & 3 \end{pmatrix}$
3. $\begin{pmatrix} 0 & 0 & -4 \\ -1 & 2 & -3 \\ 3 & -4 & 4 \end{pmatrix}$
4. $\begin{pmatrix} -2 & -2 & -2 \\ 3 & 2 & 4 \\ -1 & 0 & -2 \end{pmatrix}$
5. $\begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & -4 \\ 0 & 0 & -3 \end{pmatrix}$
6. $\begin{pmatrix} 2 & 1 & 1 \\ 2 & 1 & 1 \\ -2 & -1 & -1 \end{pmatrix}$
7. $\begin{pmatrix} -3 & 3 & 3 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix}$
8. $\begin{pmatrix} -1 & 4 & 2 \\ -1 & -3 & -3 \\ -4 & 2 & -2 \end{pmatrix}$
9. $\begin{pmatrix} 0 & -1 & 0 \\ 4 & 3 & -3 \\ 1 & -1 & -1 \end{pmatrix}$
10. $\begin{pmatrix} 1 & -3 & 2 \\ 2 & -6 & 4 \\ -3 & 9 & -6 \end{pmatrix}$
11. $\begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ 3 & 3 & -3 \end{pmatrix}$
12. $\begin{pmatrix} -4 & -2 & -1 \\ -3 & 1 & 2 \\ 1 & 3 & 3 \end{pmatrix}$

Correction détaillée de l'exercice 1 :

1.

$$\begin{aligned}
\operatorname{rg}\left(\begin{pmatrix} 1 & 2 & 1 \\ 3 & 3 & 3 \\ -1 & 0 & 5 \end{pmatrix}\right) &= \operatorname{rg}\left(\begin{pmatrix} 1 & 2 & 1 \\ 0 & -3 & 0 \\ 0 & 2 & 6 \end{pmatrix}\right) && \begin{array}{l} L_2 \leftarrow L_2 - 3L_1 \\ L_3 \leftarrow L_3 + L_1 \end{array} \\
&= \operatorname{rg}\left(\begin{pmatrix} 1 & 2 & 1 \\ 0 & -3 & 0 \\ 0 & 0 & 18 \end{pmatrix}\right) && L_3 \leftarrow 3L_3 + 2L_2 \\
&= 3
\end{aligned}$$

2.

$$\begin{aligned}
\operatorname{rg}\left(\begin{pmatrix} 1 & 4 & -3 \\ -4 & -1 & -2 \\ -1 & -2 & 3 \end{pmatrix}\right) &= \operatorname{rg}\left(\begin{pmatrix} 1 & 4 & -3 \\ 0 & 15 & -14 \\ 0 & 2 & 0 \end{pmatrix}\right) && \begin{array}{l} L_2 \leftarrow L_2 + 4L_1 \\ L_3 \leftarrow L_3 + L_1 \end{array} \\
&= \operatorname{rg}\left(\begin{pmatrix} 1 & 4 & -3 \\ 0 & 15 & -14 \\ 0 & 0 & 28 \end{pmatrix}\right) && L_3 \leftarrow 15L_3 - 2L_2 \\
&= 3
\end{aligned}$$

3.

$$\begin{aligned}
\operatorname{rg}\left(\begin{pmatrix} 0 & 0 & -4 \\ -1 & 2 & -3 \\ 3 & -4 & 4 \end{pmatrix}\right) &= \operatorname{rg}\left(\begin{pmatrix} -1 & 2 & -3 \\ 0 & 0 & -4 \\ 3 & -4 & 4 \end{pmatrix}\right) && L_1 \leftrightarrow L_2 \\
&= \operatorname{rg}\left(\begin{pmatrix} -1 & 2 & -3 \\ 0 & 0 & -4 \\ 0 & 2 & -5 \end{pmatrix}\right) && L_3 \leftarrow L_3 + 3L_1 \\
&= \operatorname{rg}\left(\begin{pmatrix} -1 & 2 & -3 \\ 0 & 2 & -5 \\ 0 & 0 & -4 \end{pmatrix}\right) && L_2 \leftrightarrow L_3 \\
&= 3
\end{aligned}$$

4.

$$\begin{aligned}
\operatorname{rg}\left(\begin{pmatrix} -2 & -2 & -2 \\ 3 & 2 & 4 \\ -1 & 0 & -2 \end{pmatrix}\right) &= \operatorname{rg}\left(\begin{pmatrix} -2 & -2 & -2 \\ 0 & -2 & 2 \\ 0 & 2 & -2 \end{pmatrix}\right) && \begin{array}{l} L_2 \leftarrow 2L_2 + 3L_1 \\ L_3 \leftarrow 2L_3 - L_1 \end{array} \\
&= \operatorname{rg}\left(\begin{pmatrix} -2 & -2 & -2 \\ 0 & -2 & 2 \\ 0 & 0 & 0 \end{pmatrix}\right) && L_3 \leftarrow L_3 + L_2 \\
&= 2
\end{aligned}$$

5.

$$\operatorname{rg}\left(\begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & -4 \\ 0 & 0 & -3 \end{pmatrix}\right) = 1$$

6.

$$\begin{aligned}
\operatorname{rg}\left(\begin{pmatrix} 2 & 1 & 1 \\ 2 & 1 & 1 \\ -2 & -1 & -1 \end{pmatrix}\right) &= \operatorname{rg}\left(\begin{pmatrix} 2 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right) && \begin{array}{l} L_2 \leftarrow L_2 - L_1 \\ L_3 \leftarrow L_3 + L_1 \end{array} \\
&= 1
\end{aligned}$$

7.

$$\begin{aligned}
\mathrm{rg}\left(\begin{pmatrix} -3 & 3 & 3 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix}\right) &= \mathrm{rg}\left(\begin{pmatrix} -3 & 3 & 3 \\ 0 & 1 & 2 \\ 0 & 3 & 6 \end{pmatrix}\right) && L_3 \leftarrow 3L_3 + L_1 \\
&= \mathrm{rg}\left(\begin{pmatrix} -3 & 3 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}\right) && L_3 \leftarrow L_3 - 3L_2 \\
&= 2
\end{aligned}$$

8.

$$\begin{aligned}
\mathrm{rg}\left(\begin{pmatrix} -1 & 4 & 2 \\ -1 & -3 & -3 \\ -4 & 2 & -2 \end{pmatrix}\right) &= \mathrm{rg}\left(\begin{pmatrix} -1 & 4 & 2 \\ 0 & -7 & -5 \\ 0 & -14 & -10 \end{pmatrix}\right) && \begin{array}{l} L_2 \leftarrow L_2 - L_1 \\ L_3 \leftarrow L_3 - 4L_1 \end{array} \\
&= \mathrm{rg}\left(\begin{pmatrix} -1 & 4 & 2 \\ 0 & -7 & -5 \\ 0 & 0 & 0 \end{pmatrix}\right) && L_3 \leftarrow L_3 - 2L_2 \\
&= 2
\end{aligned}$$

9.

$$\begin{aligned}
\mathrm{rg}\left(\begin{pmatrix} 0 & -1 & 0 \\ 4 & 3 & -3 \\ 1 & -1 & -1 \end{pmatrix}\right) &= \mathrm{rg}\left(\begin{pmatrix} 4 & 3 & -3 \\ 0 & -1 & 0 \\ 1 & -1 & -1 \end{pmatrix}\right) && L_1 \leftrightarrow L_2 \\
&= \mathrm{rg}\left(\begin{pmatrix} 4 & 3 & -3 \\ 0 & -1 & 0 \\ 0 & -7 & -1 \end{pmatrix}\right) && L_3 \leftarrow 4L_3 - L_1 \\
&= \mathrm{rg}\left(\begin{pmatrix} 4 & 3 & -3 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\right) && L_3 \leftarrow L_3 - 7L_2 \\
&= 3
\end{aligned}$$

10.

$$\begin{aligned}
\mathrm{rg}\left(\begin{pmatrix} 1 & -3 & 2 \\ 2 & -6 & 4 \\ -3 & 9 & -6 \end{pmatrix}\right) &= \mathrm{rg}\left(\begin{pmatrix} 1 & -3 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right) && \begin{array}{l} L_2 \leftarrow L_2 - 2L_1 \\ L_3 \leftarrow L_3 + 3L_1 \end{array} \\
&= 1
\end{aligned}$$

11.

$$\begin{aligned}
\mathrm{rg}\left(\begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ 3 & 3 & -3 \end{pmatrix}\right) &= \mathrm{rg}\left(\begin{pmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right) && \begin{array}{l} L_2 \leftarrow L_2 - L_1 \\ L_3 \leftarrow L_3 - 3L_1 \end{array} \\
&= 1
\end{aligned}$$

12.

$$\begin{aligned}
\mathrm{rg}\left(\begin{pmatrix} -4 & -2 & -1 \\ -3 & 1 & 2 \\ 1 & 3 & 3 \end{pmatrix}\right) &= \mathrm{rg}\left(\begin{pmatrix} -4 & -2 & -1 \\ 0 & 10 & 11 \\ 0 & 10 & 11 \end{pmatrix}\right) && \begin{array}{l} L_2 \leftarrow 4L_2 - 3L_1 \\ L_3 \leftarrow 4L_3 + L_1 \end{array} \\
&= \mathrm{rg}\left(\begin{pmatrix} -4 & -2 & -1 \\ 0 & 10 & 11 \\ 0 & 0 & 0 \end{pmatrix}\right) && L_3 \leftarrow L_3 - L_2 \\
&= 2
\end{aligned}$$