TD 06 – Sommes et Produits (Correction)

Exercice 1 – 1. $\sum_{k=1}^{20} k^3 = 1 + 2^3 + 3^3 + \ldots + 20^3$

2. $\sum_{\ell=2}^{8} \exp(\ell+1) = \exp(2+1) + \exp(3+1) + \ldots + \exp(8+1)$

3. $\sum_{i=1}^{m} (-1)^i = (-1)^1 + (-1)^2 + \cdots + (-1)^n$

4. $\sum_{j=2}^{n+1} \ln(j-1) = \ln(1) + \ln(2) + \ldots + \ln(n)$

Exercice 2 – 1. $2+4+6+8+\ldots+50 = 2\times 1 + 2\times 2 + 2\times 3 + 2\times 4 + \ldots + 2\times 25 = \sum_{k=1}^{25} 2k$

2. $1 + 2^6 + 3^6 + 4^6 + \cdots + (n+1)^6 = \sum_{k=1}^{n+1} k^6$

3. $1 - a + a^2 - a^3 + \ldots + a^{100} = \sum_{k=0}^{100} (-1)^k a^k$

Exercice 3 – 1. $\sum_{i=0}^{n} 4i = 4\sum_{i=0}^{n} i = 2n(n+1)$

2.

$$\begin{aligned}
\sum_{l=4}^{n} \frac{l-1}{4} &= \frac{1}{4}\left(\sum_{l=4}^{n} l - \sum_{l=4}^{n} 1\right) \\
&= \frac{1}{4}\left(\sum_{l=1}^{n} l - 1 - 2 - 3 - (n-4+1)\right) \\
&= \frac{1}{4}\left(\frac{n(n+1)}{2} - 6 - n + 3\right) \\
&= \frac{1}{4}\left(\frac{n(n+1) - 2(n+3)}{2}\right) \\
&= \frac{n^2 + n - 2n - 6}{8} \\
&= \frac{n^2 - n - 6}{8}
\end{aligned}$$

3.

$$\begin{aligned}
\sum_{j=1}^{n} e^{-j} &= \sum_{j=1}^{n} \left(\frac{1}{e}\right)^j \\
&= \frac{1}{e} \times \frac{1 - \left(\frac{1}{e}\right)^{n-1+1}}{1 - \frac{1}{e}} \\
&= \frac{1 - e^{-n}}{e - 1}
\end{aligned}$$

4.

$$\begin{aligned}
\sum_{l=4}^{n+1} \frac{2^l}{3^{l-2}} &= 9\sum_{l=4}^{n+1} \left(\frac{2}{3}\right)^l \\
&= 9 \times \left(\frac{2}{3}\right)^4 \times \frac{1 - \left(\frac{2}{3}\right)^{n+1-4+1}}{1 - \frac{2}{3}} \\
&= \frac{16}{9} \frac{1 - \left(\frac{2}{3}\right)^{n-2}}{\frac{1}{3}} \\
&= \frac{16}{3}\left(1 - \left(\frac{2}{3}\right)^{n-2}\right)
\end{aligned}$$

5.

$$\sum_{k=2}^{n+1} 3 = 3x(n+1-2+1) \\ = 3n$$

6.

$$S = \sum_{k=0}^{n} x^{2k+1} = x \times \sum_{k=0}^{n} \left(x^2\right)^k$$

si $x \neq 1$ $\quad S = x \times \dfrac{1-\left(x^2\right)^{n-0+1}}{1-x^2}$

$= x \times \dfrac{1-x^{2n+2}}{1-x^2}$

si $x = 1$ $\quad S = n - 0 + 1$

$= n+1$

7.

$$\sum_{k=0}^{n} \frac{3}{10^k} = 3\sum_{k=0}^{n}\left(\frac{1}{10}\right)^k$$

$$= 3 \times \left(\frac{1}{10}\right)^0 \frac{1-\left(\frac{1}{10}\right)^{n-0+1}}{1-\frac{1}{10}}$$

$$= 3 \times \frac{10}{9} \times \left(1-\left(\frac{1}{10}\right)^{n+1}\right)$$

$$= \frac{10}{3} \times \left(1 - \frac{1}{10^{n+1}}\right)$$

8.

$$\sum_{j=1}^{n-1} (5j+2-n) = 5 \times \frac{(n-1)n}{2} + (2-n)(n-1-1+1)$$

$$= 5\frac{n(n-1)}{2} - (n-1)(n-2)$$

$$= (m-1)\left[\frac{5n}{2} - n + 2\right]$$

$$= \frac{n-1}{2}[5n-2n+4]$$

$$= \frac{(m-1)(3n+4)}{2}$$

9.

$$\sum_{j=1}^{m-1} 2^j = 2^1 \times \frac{1-2^{m-1-1+1}}{1-2}$$

$$= 2 \times \left(2^{m-2} - 1\right)$$

$$= 2^n - 2$$

10.

$$\sum_{i=1}^{2N} i(2i+3) = 2x\frac{2N(2N+1)(4N+1)}{6} + 3\frac{2N(2N+1)}{2}$$

$$= \frac{2N(2N+1)}{6}[2(4N+1)+9]$$

$$= \frac{N(2N+1)(8N+11)}{3}$$

11.

$$\begin{aligned}\sum_{k=13}^{42} k &= \sum_{k=1}^{42} k - \sum_{k=1}^{12} k\\ &= \frac{42\times 43}{2} - \frac{12\times 13}{2}\\ &= 21\times 4^3 - 6\times 13\\ &= 903 - 78\\ &= 825\end{aligned}$$

12.

$$\begin{aligned}\sum_{k=0}^{n} 2^k 5^{n-k} &= 5^n \sum_{k=0}^{n}\left(\frac{2}{5}\right)^n\\ &= 5^n \times \frac{1-\left(\frac{2}{5}\right)^{n+1}}{1-\left(\frac{2}{5}\right)}\\ &= \frac{5}{3}\cdot\left(5^n - \frac{2^{n+1}}{5}\right)\\ &= \frac{1}{3}\left(5^{n+1} - 2^{n+1}\right)\end{aligned}$$

Exercice 4 – 1.

$$\begin{aligned}2+4+\ldots+100 &= 2\sum_{k=1}^{50} k\\ &= 2\frac{50\times 51}{2}\\ &= 2550\end{aligned}$$

2.

$$S = 1 - x + x^2 + \ldots + (-1)^n x^n = \sum_{k=0}^{n} (-x)^k$$

- si $x = 1$: $S = 1\times(n-0+1) = n+1$
- si $x \neq 1$: $S = \dfrac{1-(-x)^{n+1}}{1-(-x)} = \dfrac{1-(-x)^{n+1}}{1+x}$

3.

$$\begin{aligned}1+3+5+\ldots+99 &= \sum_{k=0}^{49}(2k+1)\\ &= 2\times\frac{49\times 50}{2} + (49-0+1)\\ &= 2450+50\\ &= 2500\end{aligned}$$

4.

$$\begin{aligned}2\times 5^2 + \ldots + 2\times 5^{2n+2} &= 2\sum_{k=2}^{2n+2} 5^k\\ &= 2\times 5^2\times\frac{1-5^{2n+2-2+1}}{1-5}\\ &= 50\times\frac{1-5^{2n+1}}{-4}\\ &= \frac{25}{2}\times\left(5^{2n+1}-1\right)\end{aligned}$$

Exercice 5 –

Exercice 6 – 1. Soit $x \in \mathbb{R}\setminus\{0,-1\}$. On a:

$$\begin{aligned}\frac{1}{x}-\frac{1}{x+1} &= \frac{1\times(x+1)}{x\times(x+1)}-\frac{1\times x}{(x+1)\times x}\\ &= \frac{x+1-x}{x(x+1)}\\ &= \frac{1}{x(x+1)}\end{aligned}$$

2) Soit $n \in \mathbb{N}^*$.

$$\begin{aligned}\sum_{k=1}^{n}\frac{1}{k(k+1)} &= \sum_{k=1}^{n}\left(\frac{1}{k}-\frac{1}{k+1}\right) && \textit{en utilisant la question 1}\\ &= \left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right)\\ &= 1-\frac{1}{n+1} && \textit{par télescopage}\end{aligned}$$

Exercice 7 – Soit $n \in \mathbb{N}^*$.

$$\begin{aligned}S_1 &= \sum_{k=1}^{n}(\ln k-\ln(k+1))\\ &= -\ln(n+1) && \textit{par télescopage}\end{aligned}$$

$$\begin{aligned}S_2 &= \sum_{i=0}^{n}\frac{\sqrt{i}-\sqrt{i+1}}{i-(i+1)}\\ &= \sum_{i=0}^{n}(\sqrt{i+1}-\sqrt{i})\\ &= \sqrt{n+1} && \textit{par télescopage}\end{aligned}$$

Exercice 8 – 1. Soient $k \in \mathbb{N}^*$ et a, b deux réels à déterminer. On a

$$\begin{aligned}\frac{a}{k}+\frac{b}{k+1} &= \frac{a(k+1)+bk}{k(k+1)}\\ &= \frac{(a+b)k+a}{k(k+1)}\end{aligned}$$

Donc

$$\begin{aligned}\forall k \in \mathbb{N}^\star,\quad \frac{a}{k}+\frac{b}{k+1}=\frac{1}{k(k+1)} &\Leftrightarrow \forall k \in \mathbb{N}^A, 1=(a+b)k+a\\ &\Leftrightarrow \begin{cases} a=1\\ a+b=0\end{cases}\\ &\Leftrightarrow \begin{cases} a=1\\ b=-1\end{cases}\end{aligned}$$

Donc

$$\forall k \in \mathbb{N}^\star, \frac{1}{k(k+1)}=\frac{1}{k}-\frac{1}{k+1}$$

2. Soient $n \in \mathbb{N}^*$ et $j \in \{1,\ldots,n\}$ on a :

$$\begin{aligned}
P_j &= \sum_{k=j}^{n} \frac{1}{n} \times \frac{2j}{k(k+1)} \\
&= \frac{2j}{n} \times \sum_{k=j}^{n} \frac{1}{k(k+1)} \\
&= \frac{2j}{n} \times \sum_{k=j}^{n} \left(\frac{1}{k} - \frac{1}{k+1} \right) \qquad \textit{en utilisant la question 1} \\
&= \frac{2j}{n} \times \left(\frac{1}{j} - \frac{1}{n+1} \right) \qquad \textit{par télescopage} \\
&= \frac{2j}{n} \times \frac{n+1-j}{j(n+1)} \\
&= \frac{2(n+1-j)}{n(n+1)}
\end{aligned}$$

3. Soit $n \in \mathbb{N}^*$. On a:

$$\begin{aligned}
\sum_{j=1}^{n} j \times p_j &= \sum_{j=1}^{n} j \times \frac{2(n+1-j)}{n(n+1)} \\
&= \frac{1}{n(n+1)} \sum_{j=1}^{n} 2j(n+1-j) \\
&= \frac{1}{n(n+1)} \left(\sum_{j=1}^{n} 2(n+1)j - \sum_{j=1}^{n} 2j^2 \right) \\
&= \frac{1}{n(n+1)} \left(2(n+1) \sum_{j=1}^{n} j - 2 \sum_{j=1}^{n} j^2 \right) \\
&= \frac{1}{n(n+1)} \left(2(n+1) \times \frac{n(n+1)}{2} - 2 \times \frac{n(n+1)(2n+1)}{6} \right) \\
&= \frac{1}{n(n+1)} \times 2(n+1) \times \frac{n(n+1)}{2} - \frac{1}{n(n+1)} \times 2 \times \frac{n(n+1)(2n+1)}{6} \\
&= \frac{n+1}{3} - \frac{2n+1}{3} \\
&= \frac{3(n+1)-(2n+1)}{3} \\
&= \frac{n+2}{3}
\end{aligned}$$

Exercice 9 – 1.

$$\begin{aligned}
\sum_{k=2}^{n} \frac{k+1}{k} &= \frac{3}{2} + \frac{4}{3} + \cdots + \frac{n+1}{n} \\
&= \sum_{i=3}^{n+1} \frac{i}{i-1}
\end{aligned}$$

2.

$$\begin{aligned}
\sum_{i=0}^{n} 2^i &= 2^0 + 2 + 2^2 + \cdots + 2^n \\
&= \sum_{k=1}^{n+1} 2^{k-1}
\end{aligned}$$

3.

$$\begin{aligned}\sum_{i=2}^{n}(i-2) &= 0+1+2+\ldots+n-2\\ &= \sum_{k=0}^{n-2}k\end{aligned}$$

Exercice 10 – **1.**

$$\begin{aligned}S &= \sum_{i=1}^{n}\sum_{j=1}^{p}(i+j)\\ &= \sum_{i=1}^{n}\left(i\times p+\frac{p(p+1)}{2}\right)\\ &= p\times\frac{n(n+1)}{2}+\frac{p(p+1)}{2}\times n\\ &= \frac{np}{2}[n+p+2]\end{aligned}$$

2.

$$\begin{aligned}V &= \sum_{j=1}^{n}\sum_{i=1}^{j}j\\ &= \sum_{j=1}^{n}j^2\\ &= \frac{n(n+1)(2n+1)}{6}\end{aligned}$$

3.

Si $x=1$ alors $W=\sum_{i=1}^{n}\sum_{j=1}^{n}1=n^2$

Si $x\neq 1$ alors

$$\begin{aligned}W &= \sum_{i=1}^{n}\sum_{j=1}^{n}x^{i+j}\\ &= \sum_{i=1}^{n}x^i\times\sum_{j=1}^{n}x^j\\ &= x\times\frac{1-x^n}{1-x}\times x\times\frac{1-x^n}{1-x}\\ &= \left(x\times\frac{1-x^n}{1-x}\right)^2\end{aligned}$$

Exercice 11 – **1.**

$$\begin{aligned}\prod_{i=0}^{n}2 &= \underbrace{2\times 2\times\ldots\times 2}_{n-0+1\text{ fois}}\\ &= 2^{n+1}\end{aligned}$$

2.

$$\begin{aligned}\prod_{k=1}^{n-1}2^k &= 2^1\times 2^2\times\ldots\times 2^{n-1}\\ &= 2^{1+2+\ldots+n-1}\\ &= 2^{\frac{(n-1)n}{2}} \qquad \text{car } 1+2+\ldots+n-1=\sum_{k=1}^{n-1}k=\frac{(n-1)n}{2} \quad \text{(somme finie usuelle)}\end{aligned}$$

3.

$$\begin{aligned}\prod_{k=2}^{n}3x &= \underbrace{(3x)\times(3x)\times\ldots\times(3x)}_{n-2+1\text{ fois}}\\ &= (3x)^{n-1}\end{aligned}$$

4.

$$\begin{aligned}\prod_{k=1}^{n} e^k &= e^1 \times e^2 \times \ldots \times e^n \\ &= e^{1+2+\ldots+n} \\ &= e^{n(n+1)/2} \text{ car } 1+2+\ldots+n = \sum_{k=1}^{n} k = \frac{n(n+1)}{2} \text{ (somme finie usuelle)}\end{aligned}$$

Exercice 12 – 1.

$$\begin{aligned}\prod_{k=13}^{56} \frac{k+1}{k} &= \frac{14}{13} \times \frac{15}{14} \times \frac{16}{15} \times \ldots \times \frac{56}{55} \times \frac{57}{56} \\ &= \frac{57}{13} \qquad \textit{par télescopage}\end{aligned}$$

2.

$$\begin{aligned}\prod_{k=2}^{n} \left(1 - \frac{1}{k}\right) &= \prod_{k=2}^{n} \frac{k-1}{k} \\ &= \frac{1}{2} \times \frac{2}{3} \times \cdots \times \frac{n-2}{n-1} \times \frac{n-1}{n} \\ &= \frac{1}{n} \qquad \textit{par télescopage}\end{aligned}$$

Exercice 13 – 1.

$$\begin{aligned}5 \times 6 \times \ldots \times 9 &= \frac{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9}{1 \times 2 \times 3 \times 4} \\ &= \frac{9!}{4!}\end{aligned}$$

2.

$$\begin{aligned}n(n-1)(n-2) &= \frac{1 \times 2 \times 3 \times \ldots \times (n-3) \times (n-2)(n-1)n}{1 \times 2 \times 3 \times \ldots \times (n-3)} \\ &= \frac{n!}{(n-3)!}\end{aligned}$$

3.

$$\begin{aligned}2 \times 4 \times \ldots \times (2n) &= (2 \times 1) \times (2 \times 2) \times \ldots \times (2 \times n) \\ &= 2 \times 2 \times \ldots \times 2 \times 1 \times 2 \times \ldots \times n \\ &= 2^n \times n!\end{aligned}$$

4.

$$\begin{aligned}1 \times 3 \times 5 \times \ldots \times (2n+1) &= \frac{1 \times 2 \times 3 \times 4 \times 5 \times \ldots \times (2n) \times (2n+1)}{2 \times 4 \times \ldots \times 2n} \\ &= \frac{(2n+1)!}{2^n \times n!}\end{aligned}$$

en utilisant le résultat de la question précédente

Exercice 14 – 1.

$$\begin{aligned}\frac{n!}{(n+1)!} &= \frac{1 \times 2 \times 3 \times \ldots \times (n-1) \times n}{1 \times 2 \times 3 \times \ldots \times (n-1) \times n \times (n+1)} \\ &= \frac{1}{n+1}\end{aligned}$$

2.

$$\begin{aligned}\frac{(2n+1)!}{(2n-1)!} &= \frac{1 \times 2 \times \ldots \times (2n-1) \times 2n \times (2n+1)}{1 \times 2 \times \ldots \times (2n-1)} \\ &= 2n \times (2n+1)\end{aligned}$$

3.

$$\begin{aligned}(n+1)! + (n-1)! &= (n-1)!((n+1)n+1) \\ &= (n-1)!\left(n^2+n+1\right)\end{aligned}$$

Exercice 15 – 1.

$$\begin{aligned}\prod_{j=1}^{n-1} j^2 &= 1^2 \times 2^2 \times \ldots \times (n-1)^2 \\ &= [1 \times 2 \times \ldots \times (n-1)]^2 \\ &= [(n-1)!]^2\end{aligned}$$

2.

$$\begin{aligned}\prod_{k=4}^{n} k^3 &= 4^3 \times 5^3 \times \ldots \times n^3 \\ &= (4 \times 5 \times \ldots \times n)^3 \\ &= \left(\frac{1 \times 2 \times 3 \times 4 \times 5 \times \ldots \times n}{1 \times 2 \times 3}\right)^3 \\ &= \left[\frac{n!}{3!}\right]^3\end{aligned}$$