TRAVAUX DIRIGÉS : FONCTIONS DE PLUSIEURS VARIABLES
RÉPONSES - INDICATIONS

Exercice 1.

(1) U est borné, (2) U n'est pas borné,

(3) U n'est pas borné, (4) U n'est pas borné.

Exercice 2.

(1) U est borné, (2) U n'est pas borné,

(3) U est borné, (4) U est borné.

Exercice 3.

(1) $f(x,y) = x^2 + y^2 + xy^3$:

- $\mathcal{D}_f = \mathbb{R}^2$,
- $\partial_1(f)(x,y) = 2x + y^3$,
- $\partial_2(f)(x,y) = 2y + 3xy^2$,
- $\partial^2_{1,1}(f)(x,y) = 2$,
- $\partial^2_{1,2}(f)(x,y) = \partial^2_{2,1}(f)(x,y) = 3y^2$,
- $\partial_{2,2}(f)(x,y) = 2 + 6xy$.

(2) $f(x,y) = (x^2 + y^2)\cos(y)$:

- $\mathcal{D}_f = \mathbb{R}^2$,
- $\partial_1(f)(x,y) = 2x\cos(y)$,
- $\partial_2(f)(x,y) = 2y\cos(y) - (x^2 + y^2)\sin(y)$,
- $\partial^2_{1,1}(f)(x,y) = 2\cos(y)$,
- $\partial^2_{1,2}(f)(x,y) = \partial^2_{2,1}(f)(x,y) = -2x\sin(y)$,
- $\partial_{2,2}(f)(x,y) = (2 - x^2 - y^2)\cos(y) - 4y\sin(y)$.

(3) $f(x,y) = \dfrac{\ln(x^2 + y^2)}{x^2 + y^2}$:

- $\mathcal{D}_f = \mathbb{R}^2 \setminus \{(0,0)\}$,
- $\partial_1(f)(x,y) = \dfrac{2x(1 - \ln(x^2 + y^2))}{(x^2 + y^2)^2}$,
- $\partial_2(f)(x,y) = \dfrac{2y(1 - \ln(x^2 + y^2))}{(x^2 + y^2)^2}$,
- $\partial^2_{1,1}(f)(x,y) = \dfrac{4x^2(2\ln(x^2 + y^2) - 3)}{(x^2 + y^2)^3} + \dfrac{2(1 - \ln(x^2 + y^2))}{(x^2 + y^2)^2}$,
- $\partial^2_{1,2}(f)(x,y) = \partial^2_{2,1}(f)(x,y) = \dfrac{4xy(2\ln(x^2 + y^2) - 3)}{(x^2 + y^2)^3}$,
- $\partial_{2,2}(f)(x,y) = \dfrac{4y^2(2\ln(x^2 + y^2) - 3)}{(x^2 + y^2)^3} + \dfrac{2(1 - \ln(x^2 + y^2))}{(x^2 + y^2)^2}$.

(4) $f(x,y) = xe^y + ye^x - xye^{x+y}$:

- $\mathcal{D}_f = \mathbb{R}^2$,
- $\partial_1(f)(x,y) = e^y + ye^x - (1+x)ye^{x+y}$,
- $\partial_2(f)(x,y) = xe^y + e^x - (1+y)xe^{x+y}$,
- $\partial^2_{1,1}(f)(x,y) = ye^x - y(2+x)e^{x+y}$,
- $\partial^2_{1,2}(f)(x,y) = \partial^2_{2,1}(f)(x,y) = e^x + e^y - (1+x)(1+y)e^{x+y}$,
- $\partial_{2,2}(f)(x,y) = xe^y - x(2+y)e^{x+y}$.

(5) $f(x,y) = x\sin\left(\dfrac{1}{y^2}\right)$:

- $\mathcal{D}_f = \mathbb{R} \times \mathbb{R}^*$,
- $\partial_1(f)(x,y) = \sin\left(\dfrac{1}{y^2}\right)$,
- $\partial_2(f)(x,y) = \dfrac{-2x}{y^3}\cos\left(\dfrac{1}{y^2}\right)$,
- $\partial^2_{1,1}(f)(x,y) = 0$,
- $\partial^2_{1,2}(f)(x,y) = \partial^2_{2,1}(f)(x,y) = \dfrac{-2}{y^3}\cos\left(\dfrac{1}{y^2}\right)$,
- $\partial_{2,2}(f)(x,y) = x\left(\dfrac{6}{y^4}\cos\left(\dfrac{1}{y^2}\right) - \dfrac{4}{y^6}\sin\left(\dfrac{1}{y^2}\right)\right)$.

(6) $f(x,y) = xye^{-x^2-y^2}$:

- $\mathcal{D}_f = \mathbb{R}^2$,
- $\partial_1(f)(x,y) = y(1-2x^2)e^{-x^2-y^2}$,
- $\partial_2(f)(x,y) = x(1-2y^2)e^{-x^2-y^2}$,
- $\partial^2_{1,1}(f)(x,y) = y(4x^3-6x)e^{-x^2-y^2}$,
- $\partial^2_{1,2}(f)(x,y) = \partial^2_{2,1}(f)(x,y) = (1-2x^2)(1-2y^2)e^{-x^2-y^2}$,
- $\partial_{2,2}(f)(x,y) = x(4y^3-6y)e^{-x^2-y^2}$.

(7) $f(x,y) = \dfrac{xy}{x+y}$:

- $\mathcal{D}_f = \left\{(x,y) \in \mathbb{R}^2,\ x+y \neq 0\right\}$,
- $\partial_1(f)(x,y) = \dfrac{y^2}{(x+y)^2}$,
- $\partial_2(f)(x,y) = \dfrac{x^2}{(x+y)^2}$,
- $\partial^2_{1,1}(f)(x,y) = \dfrac{-2y^2}{(x+y)^3}$,
- $\partial^2_{1,2}(f)(x,y) = \partial^2_{2,1}(f)(x,y) = \dfrac{2xy}{(x+y)^3}$,
- $\partial_{2,2}(f)(x,y) = \dfrac{-2x^2}{(x+y)^3}$.

(8) $f(x,y) = x^y$:

- $\mathcal{D}_f = \mathbb{R}_+^* \times \mathbb{R}$,
- $\partial_1(f)(x,y) = yx^{y-1}$,
- $\partial_2(f)(x,y) = \ln(x)x^y$,
- $\partial_{1,1}^2(f)(x,y) = y(y-1)x^{y-2}$,
- $\partial_{1,2}^2(f)(x,y) = \partial_{2,1}^2(f)(x,y) = (1 + y\ln(x))x^{y-1}$,
- $\partial_{2,2}(f)(x,y) = \ln^2(x)x^y$.

(9) $f(x,y) = \dfrac{x}{\sqrt{x^2+y^2}}$:

- $\mathcal{D}_f = \mathbb{R}^2 \setminus \{(0,0)\}$,
- $\partial_1(f)(x,y) = \dfrac{y^2}{(x+y)^{3/2}}$,
- $\partial_2(f)(x,y) = \dfrac{-xy}{(x+y)^{3/2}}$,
- $\partial_{1,1}^2(f)(x,y) = \dfrac{-3xy^2}{(x+y)^{5/2}}$,
- $\partial_{1,2}^2(f)(x,y) = \partial_{2,1}^2(f)(x,y) = \dfrac{y(2x^2-y^2)}{(x+y)^{5/2}}$,
- $\partial_{2,2}(f)(x,y) = \dfrac{-x(x^2-2y^2)}{(x+y)^{5/2}}$.

Exercice 4.

(1) $f(x,y,z) = x\cos(y) + y\cos(z) + z\cos(x)$:

- $\mathcal{D}_f = \mathbb{R}^3$,
- $\partial_1(f)(x,y,z) = \cos(y) - z\sin(x)$,
- $\partial_2(f)(x,y,z) = \cos(z) - x\sin(y)$,
- $\partial_3(f)(x,y,z) = \cos(x) - y\sin(z)$,
- $\partial_{1,1}^2(f)(x,y,z) = -z\cos(x)$,
- $\partial_{1,2}^2(f)(x,y,z) = \partial_{2,1}^2(f)(x,y,z) = -\sin(y)$,
- $\partial_{1,3}^2(f)(x,y,z) = \partial_{3,1}^2(f)(x,y,z) = -\sin(x)$,
- $\partial_{2,2}(f)(x,y,z) = -x\cos(y)$,
- $\partial_{2,3}^2(f)(x,y,z) = \partial_{3,2}^2(f)(x,y,z) = -\sin(z)$,
- $\partial_{3,3}(f)(x,y,z) = -y\cos(z)$.

(2) $f(x,y,z) = x\ln(y) + z\ln(x)$:

- $\mathcal{D}_f = (\mathbb{R}_+^*)^2 \times \mathbb{R}$,
- $\partial_1(f)(x,y,z) = \ln(y) + \dfrac{z}{x}$,
- $\partial_2(f)(x,y,z) = \dfrac{x}{y}$,
- $\partial_3(f)(x,y,z) = \ln(x)$,

- $\partial^2_{1,1}(f)(x,y,z) = \dfrac{-z}{x^2}$,
- $\partial^2_{1,2}(f)(x,y,z) = \partial^2_{2,1}(f)(x,y,z) = \dfrac{1}{y}$,
- $\partial^2_{1,3}(f)(x,y,z) = \partial^2_{3,1}(f)(x,y,z) = \dfrac{1}{x}$,
- $\partial_{2,2}(f)(x,y,z) = \dfrac{-x}{y^2}$,
- $\partial^2_{2,3}(f)(x,y,z) = \partial^2_{3,2}(f)(x,y,z) = 0$,
- $\partial^2_{3,3}(f)(x,y,z) = 0$.

(3) $f(x,y,z) = xy + yz + zx - \arctan(xyz)$:

- $\mathcal{D}_f = \mathbb{R}^3$,
- $\partial_1(f)(x,y,z) = y + z - \dfrac{yz}{(xyz)^2+1}$,
- $\partial_2(f)(x,y,z) = x + z - \dfrac{xz}{(xyz)^2+1}$,
- $\partial_3(f)(x,y,z) = x + y - \dfrac{xy}{(xyz)^2+1}$,
- $\partial^2_{1,1}(f)(x,y,z) = \dfrac{2xy^2z^2}{((xyz)^2+1)^2}$,
- $\partial^2_{1,2}(f)(x,y,z) = \partial^2_{2,1}(f)(x,y,z) = 1 - \dfrac{z(1-x^2y^2z^2)}{((xyz)^2+1)^2}$,
- $\partial^2_{1,3}(f)(x,y,z) = \partial^2_{3,1}(f)(x,y,z) = 1 - \dfrac{y(1-x^2y^2z^2)}{((xyz)^2+1)^2}$,
- $\partial_{2,2}(f)(x,y,z) = \dfrac{2x^2yz^2}{((xyz)^2+1)^2}$,
- $\partial^2_{2,3}(f)(x,y,z) = \partial^2_{3,2}(f)(x,y,z) = 1 - \dfrac{x(1-x^2y^2z^2)}{((xyz)^2+1)^2}$,
- $\partial^2_{3,3}(f)(x,y,z) = \dfrac{2x^2y^2z}{((xyz)^2+1)^2}$.

(4) $f(x,y,z) = \ln(x^2+y^2+z^2)$:

- $\mathcal{D}_f = \mathbb{R}^3 \setminus \{(0,0,0)\}$,
- $\partial_1(f)(x,y,z) = \dfrac{2x}{x^2+y^2+z^2}$,
- $\partial_2(f)(x,y,z) = \dfrac{2y}{x^2+y^2+z^2}$,
- $\partial_3(f)(x,y,z) = \dfrac{2z}{x^2+y^2+z^2}$,
- $\partial^2_{1,1}(f)(x,y,z) = \dfrac{2(-x^2+y^2+z^2)}{(x^2+y^2+z^2)^2}$,
- $\partial^2_{1,2}(f)(x,y,z) = \partial^2_{2,1}(f)(x,y,z) = \dfrac{-4xy}{(x^2+y^2+z^2)^2}$,
- $\partial^2_{1,3}(f)(x,y,z) = \partial^2_{3,1}(f)(x,y,z) = \dfrac{-4xz}{(x^2+y^2+z^2)^2}$,
- $\partial_{2,2}(f)(x,y,z) = \dfrac{2(x^2-y^2+z^2)}{(x^2+y^2+z^2)^2}$,
- $\partial^2_{2,3}(f)(x,y,z) = \partial^2_{3,2}(f)(x,y,z) = \dfrac{-4yz}{(x^2+y^2+z^2)^2}$,
- $\partial^2_{3,3}(f)(x,y,z) = \dfrac{2(x^2+y^2-z^2)}{(x^2+y^2+z^2)^2}$.

(5) $f(x,y,z) = e^{x^2+xy+y^2+z^2}$:

- $\mathcal{D}_f = \mathbb{R}^3$,
- $\partial_1(f)(x,y,z) = (2x+y)e^{x^2+xy+y^2+z^2}$,
- $\partial_2(f)(x,y,z) = (2y+x)e^{x^2+xy+y^2+z^2}$,
- $\partial_3(f)(x,y,z) = 2ze^{x^2+xy+y^2+z^2}$,
- $\partial^2_{1,1}(f)(x,y,z) = (2+(2x+y)^2)e^{x^2+xy+y^2+z^2}$,
- $\partial^2_{1,2}(f)(x,y,z) = \partial^2_{2,1}(f)(x,y,z) = (1+(2x+y)(x+2y))e^{x^2+xy+y^2+z^2}$,
- $\partial^2_{1,3}(f)(x,y,z) = \partial^2_{3,1}(f)(x,y,z) = 2z(2x+y)e^{x^2+xy+y^2+z^2}$,
- $\partial_{2,2}(f)(x,y,z) = (2+(2x+y)(x+2y))e^{x^2+xy+y^2+z^2}$,
- $\partial^2_{2,3}(f)(x,y,z) = \partial^2_{3,2}(f)(x,y,z) = 2z(x+2y)e^{x^2+xy+y^2+z^2}$,
- $\partial^2_{3,3}(f)(x,y,z) = (2+4z^2)e^{x^2+xy+y^2+z^2}$.

(6) $f(x,y,z) = \dfrac{xyz}{x+y+z}$:

- $\mathcal{D}_f = \{(x,y,z) \in \mathbb{R}^3,\ x+y+z \neq 0\}$,
- $\partial_1(f)(x,y,z) = \dfrac{yz(y+z)}{(x+y+z)^2}$,
- $\partial_2(f)(x,y,z) = \dfrac{xz(x+z)}{(x+y+z)^2}$,
- $\partial_3(f)(x,y,z) = \dfrac{xy(x+y)}{(x+y+z)^2}$,
- $\partial^2_{1,1}(f)(x,y,z) = \dfrac{-2yz(y+z)}{(x+y+z)^3}$,
- $\partial^2_{1,2}(f)(x,y,z) = \partial^2_{2,1}(f)(x,y,z) = \dfrac{z(2xy+xz+yz+z^2)}{(x+y+z)^3}$,
- $\partial^2_{1,3}(f)(x,y,z) = \partial^2_{3,1}(f)(x,y,z) = \dfrac{y(2xz+yz+xy+y^2)}{(x+y+z)^3}$,
- $\partial_{2,2}(f)(x,y,z) = \dfrac{-2xz(x+z)}{(x+y+z)^3}$,
- $\partial^2_{2,3}(f)(x,y,z) = \partial^2_{3,2}(f)(x,y,z) = \dfrac{x(2yz+xz+xy+x^2)}{(x+y+z)^3}$,
- $\partial^2_{3,3}(f)(x,y,z) = \dfrac{-2xy(x+y)}{(x+y+z)^3}$.

Exercice 5. Tout d'abord, $f^{-1}(\{0\})$ est la réunion des graphes des fonctions $x \longmapsto 0$ et $x \longmapsto \dfrac{-1}{x}$. De plus, $f^{-1}(\{1\})$ est le graphe de la fonction $x \longmapsto \dfrac{1-x}{x^2}$.

Exercice 6.

$$(1)\ z = \frac{1}{2}x + \frac{1}{2}y, \quad (2)\ z = \frac{1}{\sqrt{2}}, \quad (3)\ z = \frac{1}{\sqrt{2}} - \frac{1}{4\sqrt{2}}x - \frac{1}{4\sqrt{2}}y.$$

Exercice 7. On trouve que $f(\pi+h_1, \pi+h_2) = -\pi - h_1 - h_2 + \|(h_1,h_2)\|\varepsilon(h_1,h_2)$.

Exercice 8. On trouve que $f(1+h_1, -1+h_2, -1+h_3) = -h_1 - h_2 - h_3 + \|(h_1,h_2,h_3)\|\varepsilon(h_1,h_2,h_3)$.

Exercice 9.

(1) $f(x,y) = \sin(x+y) - \cos(x-y)$:

- $\partial_1(f)(x,y) = \cos(x+y) + \sin(x-y)$,
- $\partial_2(f)(x,y) = \cos(x+y) - \sin(x-y)$,
- $\partial^2_{1,1}(f)(x,y) = -\sin(x+y) + \cos(x-y)$,
- $\partial^2_{1,2}(f)(x,y) = \partial^2_{2,1}(f)(x,y) = -\sin(x+y) + \cos(x-y)$,
- $\partial^2_{2,2}(f)(x,y,z) = -\sin(x+y) + \cos(x-y)$,
- $\nabla^2(f)(0,0) = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$,
- $f(h_1,h_2) = -1 + h_1 + h_2 + \dfrac{1}{2}(h_1^2 - 2h_1h_2 + h_2^2) + \|(h_1,h_2)\|^2\varepsilon(h_1,h_2)$.

(2) $f(x,y) = \dfrac{1}{1+x^2-y^2}$:

- $\partial_1(f)(x,y) = \dfrac{-2x}{(1+x^2-y^2)^2}$,
- $\partial_2(f)(x,y) = \dfrac{2y}{(1+x^2-y^2)^2}$,
- $\partial^2_{1,1}(f)(x,y) = \dfrac{-2+6x^2+2y^2}{(1+x^2-y^2)^3}$,
- $\partial^2_{1,2}(f)(x,y) = \partial^2_{2,1}(f)(x,y) = \dfrac{-8xy}{(1+x^2-y^2)^3}$,
- $\partial^2_{2,2}(f)(x,y,z) = \dfrac{2+2x^2+6y^2}{(1+x^2-y^2)^3}$,
- $\nabla^2(f)(0,0) = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}$,
- $f(h_1,h_2) = 1 - h_1^2 + h_2^2 + \|(h_1,h_2)\|^2\varepsilon(h_1,h_2)$.

(3) $f(x,y) = e^{-x}\cos(y)$:

- $\partial_1(f)(x,y) = -e^{-x}\cos(y)$,
- $\partial_2(f)(x,y) = -e^{-x}\sin(y)$,
- $\partial^2_{1,1}(f)(x,y) = e^{-x}\cos(y)$,
- $\partial^2_{1,2}(f)(x,y) = \partial^2_{2,1}(f)(x,y) = e^{-x}\sin(y)$,
- $\partial^2_{2,2}(f)(x,y) = -e^{-x}\cos(y)$,
- $\nabla^2(f)(0,0) = \begin{pmatrix} e^{-1} & 0 \\ 0 & -e^{-1} \end{pmatrix}$,
- $f(h_1,h_2) = 1 - h_1 + \dfrac{1}{2}(e^{-1}h_1^2 - e^{-1}h_2^2) + \|(h_1,h_2)\|^2\varepsilon(h_1,h_2)$.

Exercice 10.

(1) $f(x,y,z) = \sin(x+y) + \ln(1+x+z)$:

- $\partial_1(f)(x,y,z) = \cos(x+y) + \dfrac{1}{1+x+z}$,
- $\partial_2(f)(x,y,z) = \cos(x+y)$,
- $\partial_3(f)(x,y,z) = \dfrac{1}{1+x+z}$,

- $\partial^2_{1,1}(f)(x,y,z) = -\sin(x+y) - \dfrac{1}{(1+x+z)^2}$,
- $\partial^2_{1,2}(f)(x,y,z) = \partial^2_{2,1}(f)(x,y,z) = -\sin(x+y)$,
- $\partial^2_{1,3}(f)(x,y,z) = \partial^2_{3,1}(f)(x,y,z) = -\dfrac{1}{(1+x+z)^2}$,
- $\partial^2_{2,2}(f)(x,y,z) = -\sin(x+y)$,
- $\partial^2_{2,3}(f)(x,y,z) = \partial^2_{3,2}(f)(x,y,z) = 0$,
- $\partial^2_{3,3}(f)(x,y,z) = -\dfrac{1}{(1+x+z)^2}$,
- $\nabla^2(f)(0,0,0) = \begin{pmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & -1 \end{pmatrix}$,
- $f(h_1,h_2,h_3) = 2h_1 + h_2 + h_3 - \dfrac{1}{2}(h_1^2 + h_3^2 + 2h_1h_3) + \|(h_1,h_2,h_3)\|^2\varepsilon(h_1,h_2,h_3)$.

(2) $f(x,y,z) = \dfrac{1}{1+x-y+2z}$:

- $\partial_1(f)(x,y,z) = \dfrac{-1}{(1+x-y+2z)^2}$,
- $\partial_2(f)(x,y,z) = \dfrac{1}{(1+x-y+2z)^2}$,
- $\partial_3(f)(x,y,z) = \dfrac{-2}{(1+x-y+2z)^2}$,
- $\partial^2_{1,1}(f)(x,y,z) = \dfrac{2}{(1+x-y+2z)^3}$,
- $\partial^2_{1,2}(f)(x,y,z) = \partial^2_{2,1}(f)(x,y,z) = \dfrac{-2}{(1+x-y+2z)^3}$,
- $\partial^2_{1,3}(f)(x,y,z) = \partial^2_{3,1}(f)(x,y,z) = \dfrac{4}{(1+x-y+2z)^3}$,
- $\partial^2_{2,2}(f)(x,y,z) = \dfrac{2}{(1+x-y+2z)^3}$,
- $\partial^2_{2,3}(f)(x,y,z) = \partial^2_{3,2}(f)(x,y,z) = \dfrac{4}{(1+x-y+2z)^3}$,
- $\partial^2_{3,3}(f)(x,y,z) = \dfrac{8}{(1+x-y+2z)^3}$,
- $\nabla^2(f)(0,0,0) = \begin{pmatrix} 2 & -2 & 4 \\ -2 & 2 & 4 \\ 4 & 4 & 8 \end{pmatrix}$,
- $f(h_1,h_2,h_3) = 1 - h_1 + h_2 - 2h_3 + h_1^2 + h_2^2 + 4h_3^2 - 2h_1h_2 + 4h_1h_3 + 4h_2h_3 + \|(h_1,h_2,h_3)\|^2\varepsilon(h_1,h_2,h_3)$.

Exercice 11. Calculer la dérivée de la fonction $g : t \longmapsto f(a + t(b-a))$ et montrer qu'elle est nulle sur $[0,1]$.

Exercice 12. Calculer la dérivée de la fonction $g : t \longmapsto f(a + t(b-a))$ sur $[0,1]$, puis majorer $|g'|$ à l'aide de l'inégalité de Cauchy-Schwarz, et enfin conclure avec les accroissements finis.

Exercice 13. Utiliser l'identité de Leibniz pour dériver un produit.

Exercice 14.

(1) A faire!

(2) On trouve que, pour tout $x \in \mathcal{D}_f$ et pour tout $i \in [\![1,n]\!]$: $\partial_i(f)(x) = \dfrac{-2x_i}{(x_1^2 + ... + x_n^2)^2}$.

(3) On obtient que : $f(1+h_1, ..., 1+h_n) = -\dfrac{2}{n}\displaystyle\sum_{i=1}^{n} h_i + \|h\|\varepsilon(h)$.

(4) On trouve que $f'_u(a) = -2$ et $f''_u(a) = 8 - 2n$.

Exercice 15.

(1)
- $\nabla(f)(x,y) = (3x^2 - y, 3y^2 - x)$,
- $\mathcal{C} = \left\{(0,0), \left(\frac{1}{3}, \frac{1}{3}\right)\right\}$.

(2)
- $\nabla(f)(x,y) = (2x + 4y, 4x + 2y)$,
- $\mathcal{C} = \{(0,0)\}$.

(3)
- $\nabla(f)(x,y) = \left(\left(\frac{2x^2}{1+x^2+y^2} + \ln(1+x^2+y^2)\right) e^{x\ln(1+x^2+y^2)}, \frac{2xy}{1+x^2+y^2} e^{x\ln(1+x^2+y^2)}\right)$,
- $\mathcal{C} = \{(0,0)\}$.

(4)
- $\nabla(f)(x,y) = (4x^3 - 4x + 4y, 4y^3 - 4y + 4x)$,
- $\mathcal{C} = \left\{(0,0), \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\right\}$.

Exercice 16.

(1)
- $\nabla(f)(x,y,z) = \left(\frac{2x}{1+y^2}, -\frac{x^2+z^2}{(1+y^2)^2}, \frac{2z}{1+y^2}\right)$,
- $\mathcal{C} = \{(0,y,0),\ y \in \mathbb{R}\}$.

(2)
- $\nabla(f)(x,y,z) = (y + z - 1, x + z - 1, x + y - 1)$,
- $\mathcal{C} = \left\{\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)\right\}$.

(3)
- $\nabla(f)(x,y,z) = (2x - 2y, -2y + z + 1, y - 1)$,
- $\mathcal{C} = \{(1,1,1)\}$.

(4)
- $\nabla(f)(x,y,z) = (y + z - yz, x + z - xz, x + y - xy)$,
- $\mathcal{C} = \{(0,0,0), (2,2,2)\}$.

Exercice 17.

(1) A faire! On trouve que $f^{-1}(\{1\}) = \{(0,0)\}$.
(2) Vérifier que $\mathcal{C} = \{(0,0)\}$.
(3) Etablir que f admet un minimum global en $(0,0)$.

Exercice 18. Soit $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ l'application définie pour tout $(x,y) \in \mathbb{R}^2$ par $f(x,y) = 3x^4 - 4x^2y + y^2$.

(1) A faire! On trouve que : $f^{-1}(\{0\}) = \{(x,x^2),\ x \in \mathbb{R}\} \cup \{(x,3x^2),\ x \in \mathbb{R}\}$.
(2) Vérifier que $\mathcal{C} = \{(0,0)\}$.
(3) Montrer que $f_1(t,0) = 3t^4 \geq 0$ et conclure.
(4) Montrer que $f_2(t,2t^2) = -t^4 \geq 0$ et conclure.
(5) Etablir que f n'admet pas d'extremum local/global.

Exercice 19.

(1) A faire! On trouve que : $\nabla(f)(x_1,...,x_n) = \left(2nx_1 - \sum_{k=1}^{n} x_k, ..., 2nx_n - \sum_{k=1}^{n} x_k\right)$.
(2) On obtient que $\mathcal{C} = \{(t,...,t),\ t \in \mathbb{R}\}$.
(3) Utiliser l'inégalité de Cauchy-Schwarz.
(4) Conclure avec les questions (2) et (3).

Exercice 20.

(1) On obtient que : $\mathcal{D} = \left\{(x,y) \in \mathbb{R}^2,\ y^2 - 1 \neq 0,\ 1 + xy \neq 0,\ \frac{x+y}{1+xy} > 0\right\}$.

(2) Faire le calcul!

(3) Pour y fixé dans $]0,1[$, étudier la fonction $x \longmapsto \ln\left(\dfrac{x+y}{1+xy}\right)$ sur $\mathbb{R}_+$.

(4) Pour $x \geq 0$ fixé, montrer que l'intégrale $\displaystyle\int_0^1 f(x,y)dy$ converge absolument en utilisant la question (3) et par comparaison avec l'intégrale convergente $\displaystyle\int_0^1 \ln(y)dy$.

1. Exercices supplémentaires

Exercice 21.

(1) $f(x,y) = xy + e^x$:

- $\partial_1(f)(x,y) = y + e^x$,
- $\partial_2(f)(x,y) = x$,
- $\partial^2_{1,1}(f)(x,y) = e^x$,
- $\partial^2_{1,2}(f)(x,y) = \partial^2_{2,1}(f)(x,y) = 1$,
- $\partial^2_{2,2}(f)(x,y) = 0$,
- $\nabla(f)(0,0) = (1,0)$, $\nabla^2(f)(0,0) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$,
- $f(h_1,h_2) = 1 + h_1 + \dfrac{1}{2}h_1^2 + h_1h_2 + \|(h_1,h_2)\|^2\varepsilon(h_1,h_2)$.

(2) $f(x,y,z) = \dfrac{x}{y} + \dfrac{y}{z} + \dfrac{z}{x}$:

- $\partial_1(f)(x,y,z) = \dfrac{1}{y} - \dfrac{z}{x^2}$,
- $\partial_2(f)(x,y,z) = \dfrac{1}{z} - \dfrac{x}{y^2}$,
- $\partial_3(f)(x,y,z) = \dfrac{1}{x} - \dfrac{y}{z^2}$,
- $\partial^2_{1,1}(f)(x,y,z) = \dfrac{2z}{x^3}$,
- $\partial^2_{1,2}(f)(x,y,z) = \partial^2_{2,1}(f)(x,y,z) = \dfrac{-1}{y^2}$,
- $\partial^2_{1,3}(f)(x,y,z) = \partial^2_{3,1}(f)(x,y,z) = \dfrac{-1}{x^2}$,
- $\partial^2_{2,2}(f)(x,y,z) = \dfrac{2x}{y^3}$,
- $\partial^2_{2,3}(f)(x,y,z) = \partial^2_{3,2}(f)(x,y,z) = \dfrac{-1}{z^2}$,
- $\partial^2_{3,3}(f)(x,y,z) = \dfrac{2y}{z^3}$,
- $\nabla(f)(1,1,1) = (0,0,0)$, $\nabla^2(f)(1,1,1) = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$,
- $f(1+h_1, 1+h_2, 1+h_3) = 3 + h_1^2 + h_2^2 + h_3^2 - h_1h_2 - h_2h_3 - h_1h_3 + \|(h_1,h_2,h_3)\|^2\varepsilon(h_1,h_2,h_3)$.

(3) $f(x,y) = e^{-x}\cos(y)$:

- $\partial_1(f)(x,y) = -e^{-x}\cos(y)$,
- $\partial_2(f)(x,y) = -e^{-x}\sin(y)$,

- $\partial^2_{1,1}(f)(x,y) = e^{-x}\cos(y)$,
- $\partial^2_{1,2}(f)(x,y) = \partial^2_{2,1}(f)(x,y) = e^{-x}\sin(y)$,
- $\partial^2_{2,2}(f)(x,y) = -e^{-x}\cos(y)$,
- $\nabla(f)(0,\pi) = (1,0)$, $\nabla^2(f)(0,\pi) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$,
- $f(h_1, \pi + h_2) = -1 + h_1 - \frac{1}{2}h_1^2 + \frac{1}{2}h_2^2 + \|(h_1,h_2)\|^2\varepsilon(h_1,h_2)$.

(4) $f(x,y,z) = xy + yz + zx$:

- $\partial_1(f)(x,y,z) = y + z$,
- $\partial_2(f)(x,y,z) = x + z$,
- $\partial_3(f)(x,y,z) = x + y$,
- $\partial^2_{1,1}(f)(x,y,z) = 0$,
- $\partial^2_{1,2}(f)(x,y,z) = \partial^2_{2,1}(f)(x,y,z) = 1$,
- $\partial^2_{1,3}(f)(x,y,z) = \partial^2_{3,1}(f)(x,y,z) = 1$,
- $\partial^2_{2,2}(f)(x,y,z) = 0$,
- $\partial^2_{2,3}(f)(x,y,z) = \partial^2_{3,2}(f)(x,y,z) = 1$,
- $\partial^2_{3,3}(f)(x,y,z) = 0$,
- $\nabla(f)(1,2,3) = (5,4,3)$, $\nabla^2(f)(1,2,3) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$,
- $f(1+h_1, 2+h_2, 3+h_3) = 11 + 5h_1 + 4h_2 + 3h_3 + h_1h_2 + h_2h_3 + h_1h_3 + \|(h_1,h_2,h_3)\|^2\varepsilon(h_1,h_2,h_3)$.

Exercice 22.

(1) On trouve que $\nabla(f)(x,y) = (2x - 2y, -2x - 3y^2)$ et $\nabla^2(f)(x,y) = \begin{pmatrix} 2 & -2 \\ -2 & -6y \end{pmatrix}$.

(2) Vérifier que $\mathcal{C} = \left\{(0,0), \left(-\frac{2}{3}, -\frac{2}{3}\right)\right\}$.

(3) On obtient que $f(t,0) = t^2$ et $f(0,t) = -t^3$ pour tout $t > 0$. Pas d'extremum global.

(4) Si $\mathcal{S}$ est l'ensemble des solutions de l'équation $f(x,y) = 0$, alors on trouve que :

$$\mathcal{S} = \left\{\left(y - y\sqrt{1+y}, y\right),\ y \in [-1, +\infty[\right\} \cup \left\{\left(y + y\sqrt{1+y}, y\right),\ y \in [-1, +\infty[\right\}.$$

(5) A faire à l'aide de la question (3).

Exercice 23.

(1) A faire! On trouve que $\partial_i(f)(x_1, ..., x_n) = ix_1...x_i^{i-1}...x_n^n$ pour tout $i \in [\![1, n]\!]$.

(2) On obtient que $\nabla(f)(1, ..., 1) = (1, 2, ..., n)$.

(3) Etablir que $f(a+h) = 1 + \sum_{k=1}^{n} kh_k + \|h\|\varepsilon(h)$.

Exercice 24.

(1) A faire! On trouve que $\nabla(f)(x,y,z) = (4x - 2y - 2z, -2x + 4y - 2z, -2x - 2y + 4z)$.

(2) Vérifier que $\mathcal{C} = \{(t,t,t),\ t \in \mathbb{R}\}$.

(3) Etablir que f admet un minimum global le long de $\mathcal{C}$.

Exercice 25.

(1) A faire!
(2) On trouve que, pour tout point $(x,y) \in \mathbb{R}^2$:

$$\begin{cases} \nabla(f)(x,y) = \big(4(x+y)^3 + 4(x-y)^3 - 4(x-y), 4(x+y)^3 - 4(x-y)^3 + 4(x-y)\big) \\ \\ \nabla^2(f)(x,y) = \begin{pmatrix} 12(x+y)^2 + 12(x-y)^2 - 4 & 12(x+y)^2 - 12(x-y)^2 + 4 \\ 12(x+y)^2 - 12(x-y)^2 + 4 & 12(x+y)^2 + 12(x-y)^2 - 4 \end{pmatrix} \end{cases}.$$

(3) Vérifier que $\mathcal{C} = \left\{(0,0), \left(-\frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, -\frac{1}{2}\right)\right\}$.
(4) On trouve que $f(x,x) = 16x^4 - 8x^2$ pour tout $x \in \mathbb{R}$. La fonction f n'admet pas de maximum global.
(5) (a) Ecrire que $f(x,y) = (x+y)^4 + \big((x-y)^2 - 1\big)^2 - 1$ et conclure.
(b) Montrer que f admet un minimum global atteint aux deux derniers points critiques et valant -1.

Exercice 26.

(1) (a) Vérifier que $x^2 + y^2 \geq 2|xy|$ et conclure.
(b) Par encadrement, établir que $f(x,y)$ tend vers 0 quand (x,y) tend vers $(0,0)$.
(2) A faire! On trouve que, pour tout $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$:

$$\partial_1(f)(x,y) = \frac{y^3}{(x^2+y^2)^{3/2}} \quad \text{et} \quad \partial_2(f)(x,y) = \frac{x^3}{(x^2+y^2)^{3/2}}.$$

(3) Vérifier que $\mathcal{C} = \emptyset$.
(4) On trouve que $f(t,t) = t$ et $f(t,-t) = -t$ pour tout $t > 0$. Pas d'extremum global pour f.

Exercice 27.

(1) (a) Dériver l'application $x_i \longmapsto f(tx_1, ..., tx_n)$ pour $x_1, ..., x_{i-1}, x_{i+1}, ..., x_n, t$ fixés.
(b) Dériver l'application $t \longmapsto f(tx_1, ..., tx_n)$ pour $x_1, ..., x_n$ fixés.
(2) (a) Faire le calcul!
(b) On trouve que $g'(t) = 0$.
(c) A faire avec les questions précédentes.

Exercice 28.

(1) On trouve que $D = \big\{(x,t) \in \mathbb{R}^2,\ t \neq 0,\ t \neq -1,\ 1 + xt > 0\big\}$.
(2) Vérifier que $a_x = -\dfrac{x}{1-x}$ et $b_x = \dfrac{1}{1-x}$.
(3) Etablir que $\Delta =]-1, +\infty[$.
(4) On trouve que $F'(x) = -\dfrac{\ln(1+x)}{1-x} + \dfrac{\ln(2)}{1-x}$ pour tout $x > -1$.
(5) Vérifier que F est croissante sur $]-1,1[$ et sur $]1,+\infty[$. Pour la continuité en 1, montrer à l'aide des accroissements finis que $F'(x)$ tend vers $\ln'(2) = 1/2$ quand x tend vers 1, et conclure.
(6) Utiliser les accroissements finis pour montrer que $F'(x) \geq \dfrac{1}{1+x}$ pour tout $x > 1$. En déduire par croissance de l'intégrale que $F(x)$ tend vers $+\infty$ quand x tend vers $+\infty$.

Exercice 29.

(1) Cf cours de deuxième année.
(2) (a)
- La fonction $f_1 + f_2$ est convexe (à montrer),
- La fonction αf_1 n'est pas nécessairement convexe (prendre $f_1 : t \longmapsto t$ et $\alpha = -1$),
- La fonction $\min(f_1, f_2)$ n'est pas nécessairement convexe (prendre $f_1 : t \longmapsto t$ et $f_2 : t \longmapsto 0$),
- La fonction $\max(f_1, f_2)$ est convexe (à montrer).

(b) Si $n = 1$, la fonction $f_1 \circ f_2$ n'est pas nécessairement convexe (prendre $f_1 : t \longmapsto e^t$ et $f_2 : t \longmapsto e^{-t}$),
(3) (a) A faire!
(b) On trouve que $g'_{x,h}(t) = \displaystyle\sum_{i=1}^{n} \partial_i(f)(x+th)h_i$.
(c) Comme $g_{x,h}$ est convexe, la fonction $g'_{x,h}$ est croissante sur $[0,1]$, et donc :

$$\forall t \in [0,1], \quad g'_{x,h}(t) \geq \langle \nabla(f)(x), h \rangle.$$

En remplaçant h par $y - x$, ceci nous donne que :

$$\forall t \in [0,1], \quad g'_{x,y-x}(t) \geq \langle \nabla(f)(x), y - x \rangle.$$

Il suffit alors d'intégrer le tout sur $[0,1]$ et d'utiliser la croissance de l'intégrale pour conclure.
(d) Utiliser la question (3)(c)!

(4) (a) Exprimer $f(x)$ en fonction des coefficients $a_{i,j}$ de A et de $x_1, ..., x_n$, puis dériver partiellement.
(b) D'après (4)(a), $(0, ..., 0)$ est un point critique de f. Donc f admet un minimum global, qui est atteint au point $(0, ..., 0)$. Comme $f(0, ..., 0) = 0$, on voit que $f(x) = \frac{1}{2}\langle AX, X\rangle \geq 0$ pour tout $x \in \mathbb{R}^n$. On en déduit alors que les valeurs propres de A sont toutes positives.

Exercice 30.

(1) (a) A faire!
(b) A faire par composition!
(c) Pour tout $(x, y) \in \Omega$, on trouve que : $\nabla(g)(x, y) = \left(\frac{xf'(\sqrt{x^2+y^2})}{\sqrt{x^2+y^2}}, \frac{yf'(\sqrt{x^2+y^2})}{\sqrt{x^2+y^2}}\right)$.

(2) (a) Pour tout $(x, y) \in \Omega$, on trouve que : $\Delta(g)(x, y) = f''(\sqrt{x^2+y^2}) + \frac{f'(\sqrt{x^2+y^2})}{\sqrt{x^2+y^2}}$.
(b) Utiliser la question (2)(a)!

(3) (a) Pour tout $t > 0$, on trouve que $\varphi'(t) = f'(t) + tf''(t)$.
(b) Etablir que g est solution de l'équation $\Delta(g)(x, y) = 0$ pour tout $(x, y) \in \Omega$ si et seulement s'il existe des réels α, β tels que $g(x, y) = \alpha \ln(x^2 + y^2) + \beta$ pour tout $(x, y) \in \Omega$.
(c) A la question précédente, on voit que $g : (x, y) \longmapsto \frac{\ln(x^2+y^2)}{\ln(2)}$ convient!