IE complexes

jeudi 2 octobre 2025 15:59

1) $a = 1$, $b = 1$ et $c = i$

$\Delta = b^2 - 4ac$

$\Delta = 1 - 4i$

On cherche $\delta \in \mathbb{C}$ tel que $\delta^2 = 1 - 4i$

$\exists (x,y) \in \mathbb{R}^2$ tels que $\delta = x + iy$

$\delta^2 = x^2 - y^2 + 2ixy = 1 - 4i$

Par identification : $\begin{cases} x^2 - y^2 = 1 \\ 2xy = -4 \end{cases}$

De plus : $|\delta^2| = |1 - 4i|$

$|\delta|^2 = \sqrt{1+16}$

$x^2 + y^2 = \sqrt{17}$

Ainsi $\begin{cases} x^2 - y^2 = 1 & (1) \\ x^2 + y^2 = \sqrt{17} & (2) \\ xy = -2 < 0 & (3) \end{cases}$

(1) + (2) donne : $x^2 = \dfrac{1+\sqrt{17}}{2}$ et $x = \pm\sqrt{\dfrac{1+\sqrt{17}}{2}}$

(2) − (1) donne $y^2 = \dfrac{\sqrt{17}-1}{2}$ et $y = \pm\sqrt{\dfrac{\sqrt{17}-1}{2}}$

$xy < 0$ donc $\delta = \sqrt{\dfrac{1+\sqrt{17}}{2}} - i\sqrt{\dfrac{\sqrt{17}-1}{2}}$ ou $\delta = -\sqrt{\dfrac{1+\sqrt{17}}{2}} + i\sqrt{\dfrac{\sqrt{17}-1}{2}}$

et $z = \dfrac{-1 \pm \delta}{2}$

Donc $\boxed{S = \left\{ \dfrac{1}{2}\left[-1 + \sqrt{\dfrac{\sqrt{17}+1}{2}} - i\sqrt{\dfrac{\sqrt{17}-1}{2}}\right] ; \dfrac{1}{2}\left[-1 - \sqrt{\dfrac{1+\sqrt{17}}{2}} + i\sqrt{\dfrac{\sqrt{17}-1}{2}}\right] \right\}}$

2) On cherche $\delta \in \mathbb{C}$ tel que $\delta = x + iy$ avec $(x,y) \in \mathbb{R}^2$ et

$\delta^2 = 3 + 4i$ d'où : $x^2 - y^2 + 2ixy = 3 + 4i$

Par identification : $\begin{cases} x^2 - y^2 = 3 \\ 2xy = 4 \end{cases}$

et $|\delta^2| = |3 + 4i|$

$|\delta|^2 = \sqrt{9+16}$

$x^2 + y^2 = 5$

Finalement : $\begin{cases} x^2 - y^2 = 3 & (1) \\ x^2 + y^2 = 5 & (2) \\ xy = 2 > 0 & (3) \end{cases}$

(1) + (2) donne : $x^2 = 4$ donc $x = \pm 2$

(2) − (1) donne : $y^2 = 1$ donc $y = \pm 1$

(3) donne $xy > 0$ et $\boxed{\delta = 2 + i \text{ ou } \delta = -2 - i}$

3) $e^{ip} + e^{iq} = e^{i\left(\frac{p+q}{2}\right)}\left(e^{i\left(\frac{p-q}{2}\right)} + e^{-i\left(\frac{p-q}{2}\right)}\right)$

$\boxed{e^{ip} + e^{iq} = 2\cos\left(\frac{p-q}{2}\right) e^{i\left(\frac{p+q}{2}\right)}}$

4) a) $|1+i| = \sqrt{2}$ et $1 + i = \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) = \sqrt{2}\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)$

$\boxed{1+i = \sqrt{2}e^{\frac{i\pi}{4}}}$

$|1-i| = \sqrt{2}$ et $1 - i = \sqrt{2}\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right) = \sqrt{2}e^{-\frac{i\pi}{4}}$

$\boxed{1-i = \sqrt{2}e^{-\frac{i\pi}{4}}}$

b) $Z = \dfrac{\sqrt{2} + 1 + i}{(1-i) - \sqrt{2}} = \dfrac{\sqrt{2} + \sqrt{2}e^{\frac{i\pi}{4}}}{\sqrt{2}e^{-\frac{i\pi}{4}} - \sqrt{2}} = \dfrac{1 + e^{\frac{i\pi}{4}}}{-1 + e^{-\frac{i\pi}{4}}}$

$Z = \dfrac{e^{\frac{i\pi}{8}}\left(e^{-\frac{i\pi}{8}} + e^{\frac{i\pi}{8}}\right)}{e^{-\frac{i\pi}{8}}\left(-e^{\frac{i\pi}{8}} + e^{-\frac{i\pi}{8}}\right)} = \dfrac{2\cos\left(\frac{\pi}{8}\right)}{-2i\sin\left(\frac{\pi}{8}\right)} e^{\frac{i\pi}{4}}$

$Z = \dfrac{-\cos\left(\frac{\pi}{8}\right)}{\sin\left(\frac{\pi}{8}\right)} \times e^{\frac{i\pi}{4}} \times e^{-\frac{i\pi}{2}}$

$\boxed{Z = -\dfrac{\cos\left(\frac{\pi}{8}\right)}{\sin\left(\frac{\pi}{8}\right)} e^{-\frac{i\pi}{4}}}$

c) $\boxed{|Z| = \dfrac{\cos\left(\frac{\pi}{8}\right)}{\sin\left(\frac{\pi}{8}\right)}}$ et $\boxed{\arg Z = \pi - \frac{\pi}{4} = \frac{3\pi}{4}}$

5) Soit $z = x + iy$ avec $(x,y) \in \mathbb{R}^2$

$\dfrac{z-1}{z+1} = \dfrac{(x-1) + iy}{(x+1) + iy} = \dfrac{[(x-1) + iy][(x+1) - iy]}{(x+1)^2 + y^2}$

$= \dfrac{[x^2 - 1 + y^2] + i[y(x+1) - y(x-1)]}{(x+1)^2 + y^2}$

$\dfrac{z-1}{z+1} \in \mathbb{R} \iff y(x+1) - y(x-1) = 0$

$\iff \cancel{yx} + y - \cancel{yx} + y = 0$

$\iff y = 0$

$\dfrac{z-1}{z+1} \in \mathbb{R} \iff z \in \mathbb{R}$, et $z + 1 \neq 0$ donc $\boxed{S = \mathbb{R} \setminus \{-1\}}$

6) $|z-1|^2 = |z-i|^2 \iff (z-1)(\bar{z}-1) = (z-i)(\bar{z}+i)$

$\iff \cancel{z\bar{z}} - \bar{z} - z + \cancel{1} = \cancel{z\bar{z}} - i\bar{z} + iz + \cancel{1}$

On pose $z = x + iy$ avec $(x,y) \in \mathbb{R}^2$

$-(x - iy) - (x + iy) = -i(x - iy) + i(x + iy)$

$-x + \cancel{iy} - x - \cancel{iy} = -\cancel{ix} - y + \cancel{ix} - y$

$-2x = -2y$

$\boxed{S = \{ z \in \mathbb{C} \;/\; \mathrm{Im}(z) = \mathrm{Re}(z) \}}$