Corrigé des exercices

Exercice n°1

13

On résout par la méthode du pivot :

$$\begin{cases} -x+2y+z=4 \\ 2x-3y+4z=2 \\ 3x+4y-z=1 \end{cases} \iff \begin{cases} -x+2y+z=4 \\ \quad y+6z=10 & \leftarrow 2(1)+(2) \\ \quad 10y+2z=13 & \leftarrow 3(1)+(3) \end{cases}$$

$$\iff \begin{cases} -x+2y+z=4 \\ \quad y+6z=10 \\ \quad 58z=87 & \leftarrow 10(2)-(3) \end{cases}$$

D'où $z=\dfrac{87}{58}=\dfrac{3}{2}$, puis $y=1$ et $x=-\dfrac{1}{2}$.

$$\boxed{\mathcal{S}=\left\{\left(-\tfrac{1}{2}\,;\,1\,;\,\tfrac{3}{2}\right)\right\}}$$

Exercice n°2

13

1)

$$\sum_{i=1}^{n} 2^i = 2\times\frac{1-2^n}{1-2} = 2^{n+1}-2.$$

2) On pose $l=i+1$:

$$\begin{aligned}
\sum_{i=0}^{n}\binom{n}{i+1}2^i &= \sum_{l=1}^{n+1}\binom{n}{l}2^{l-1} = \frac{1}{2}\sum_{l=1}^{n+1}\binom{n}{l}2^l \\
&= \frac{1}{2}\left(\sum_{l=0}^{n}\binom{n}{l}2^l - \binom{n}{0}2^0 + \underbrace{\binom{n}{n+1}}_{=0}2^{n+1}\right) \\
&= \frac{1}{2}\sum_{l=0}^{n}\binom{n}{l}2^l - \frac{1}{2} \\
&= \frac{1}{2}\times(1+2)^n - \frac{1}{2} \\
&= \frac{3^n-1}{2}.
\end{aligned}$$

Exercice n°3 *15*

On note $a_n = \dfrac{1}{n+1}\binom{2n}{n}$, $S_n = \displaystyle\sum_{k=0}^{n} a_k\, a_{n-k}$ et $T_n = \displaystyle\sum_{k=0}^{n} k\, a_k\, a_{n-k}$.

1) On pose $l = n - k$:

$$T_n = \sum_{k=0}^{n} k\, a_k\, a_{n-k} = \sum_{l=n}^{0} (n-l)\, a_{n-l}\, a_l.$$

$$\boxed{T_n = \sum_{l=0}^{n} (n-l)\, a_{n-l}\, a_l} \quad (1{,}5)$$

2)

$$2T_n = \sum_{k=0}^{n} k\, a_k\, a_{n-k} + \sum_{k=0}^{n} (n-k)\, a_k\, a_{n-k}$$

$$2T_n = \sum_{k=0}^{n} (n-k+k)\, a_k\, a_{n-k}$$

$$\boxed{2T_n = n\, S_n} \quad (1{,}5)$$

3) On a

$$(n+2)a_{n+1} = \frac{n+2}{n+2}\binom{2n+2}{n+1} = \frac{(2n+2)!}{(n+1)!\,\big(2n+2-(n+1)\big)!} = \frac{(2n+2)!}{(n+1)!\,(n+1)!}$$

$$(n+2)a_{n+1} = \frac{(2n+2)(2n+1)\times(2n)!}{(n+1)(n+1)\times(n!)^2} = \frac{2(n+1)(2n+1)(2n)!}{(n+1)(n+1)(n!)^2}$$

$$(n+2)a_{n+1} = \frac{2(2n+1)(2n)!}{(n+1)(n!)^2}.$$

D'autre part,

$$2(2n+1)a_n = \frac{2(2n+1)}{n+1}\binom{2n}{n} = \frac{2(2n+1)}{n+1} \times \frac{(2n)!}{(n!)^2}.$$

D'où

$$\boxed{(n+2)\, a_{n+1} = 2(2n+1)\, a_n} \quad (2)$$

4)

$$\begin{aligned} T_{n+1} + S_{n+1} &= \sum_{k=0}^{n+1} k\, a_k\, a_{n+1-k} + \sum_{k=0}^{n+1} a_k\, a_{n+1-k} \\ &= \sum_{k=0}^{n+1} (k+1)\, a_k\, a_{n+1-k} \\ &= a_{n+1} + \underbrace{\sum_{k=1}^{n+1} (k+1)\, a_k\, a_{n+1-k}}_{\text{on pose } l=k-1} \\ &= a_{n+1} + \sum_{l=0}^{n} (l+2)\, a_{l+1}\, a_{n-l}. \end{aligned}$$

Or $(l+2)a_{l+1} = 2(2l+1)a_l$, donc

$$\begin{aligned} T_{n+1} + S_{n+1} &= a_{n+1} + 2\sum_{l=0}^{n}(2l+1)\, a_l\, a_{n-l} \\ &= a_{n+1} + 4\underbrace{\sum_{l=0}^{n} l\, a_l\, a_{n-l}}_{T_n} + 2\underbrace{\sum_{l=0}^{n} a_l\, a_{n-l}}_{S_n} \\ &= a_{n+1} + 4T_n + 2S_n \\ &= a_{n+1} + 2nS_n + 2S_n. \end{aligned}$$

$$\boxed{T_{n+1} + S_{n+1} = a_{n+1} + 2(n+1)\, S_n}\ (1)$$

Exercice n°4 *15*

1) On a $a = 1$, $b = -5$, $c = 4 + 10i$, d'où

$$\Delta = 25 - 4 \times (4 + 10i) = 9 - 40i.$$

On cherche $\delta = x + iy$ avec $(x, y) \in \mathbb{R}^2$ tel que $\delta^2 = \Delta$:

$$x^2 - y^2 + 2ixy = 9 - 40i \iff \begin{cases} x^2 - y^2 = 9 \\ 2xy = -40 \end{cases}$$

De plus $|\delta^2| = \sqrt{81 + 1600} = \sqrt{1681} = 41$, donc $x^2 + y^2 = 41$. Ainsi

$$\begin{cases} x^2 - y^2 = 9 \\ x^2 + y^2 = 41 \\ xy < 0 \end{cases}$$

d'où

$$\left.\begin{array}{rr} 2x^2 = 50 & 2y^2 = 32 \\ x^2 = 25 & y^2 = 16 \\ x = \pm 5 & y = \pm 4 \end{array}\right\} \quad \delta = 5 - 4i \ \text{ ou } \ -5 + 4i\ (2)$$

Donc

$$z = \frac{5 + 5 - 4i}{2} \quad \text{ou} \quad z = \frac{5 - 5 + 4i}{2},$$

$$\boxed{z = 5 - 2i \ \text{ ou } \ z = 2i}\ (2)$$

2) On a

$$\frac{1 + iz}{1 - iz} = \frac{(1 + iz)\,\overline{(1 - iz)}}{(1 - iz)\,\overline{(1 - iz)}} = \frac{1 - |z|^2 + i(z + \bar{z})}{|1 - iz|^2}.$$

Ce quotient est réel si et seulement si $z + \bar{z} = 0$, *i.e.* si et seulement si $\mathrm{Re}(z) = 0$. Donc

$$\boxed{\mathcal{S} = i\mathbb{R} \setminus \{-i\}}\ (1)$$

Exercice n°5 *15*

Soit $z = e^{2i\pi/5}$.

1)

$$\begin{aligned} x^2 + x - 1 &= \left(z + \frac{1}{z}\right)^2 + \left(z + \frac{1}{z}\right) - 1 \\ &= z^2 + 2 + \frac{1}{z^2} + z + \frac{1}{z} - 1 \\ &= \frac{z^4 + z^2 + 1 + z^3 + z}{z^2} \\ &= \frac{z^4 + z^3 + z^2 + z + 1}{z^2}. \end{aligned}$$

Donc, pour tout z solution de $z^4 + z^3 + z^2 + z + 1 = 0$, le nombre $x = z + \dfrac{1}{z}$ est solution de $x^2 + x - 1 = 0$. ②

2) On a

$$1 + z + \cdots + z^4 = \frac{1 - z^5}{1 - z} \qquad \text{or} \qquad \left(e^{2i\pi/5}\right)^5 = 1.$$

Donc

$$\boxed{\text{pour } z = e^{2i\pi/5}, \quad z^4 + z^3 + z^2 + z + 1 = 0}\ ②$$

3) D'après 1), $e^{2i\pi/5} + \dfrac{1}{e^{2i\pi/5}}$ est solution de $x^2 + x - 1 = 0$. Or

$$e^{2i\pi/5} + \frac{1}{e^{2i\pi/5}} = 2\cos\left(\frac{2\pi}{5}\right).$$

Donc $2\cos\left(\dfrac{2\pi}{5}\right)$ est solution de $x^2 + x - 1 = 0$. Or $\Delta = 1 - 4 \times (-1) = 5$, et $x = \dfrac{-1 \pm \sqrt{5}}{2}$. Donc

$$2\cos\left(\frac{2\pi}{5}\right) = \frac{-1 - \sqrt{5}}{2} \quad \text{ou} \quad 2\cos\left(\frac{2\pi}{5}\right) = \frac{-1 + \sqrt{5}}{2}.$$

Or $2\cos\left(\dfrac{2\pi}{5}\right) > 0$, donc $2\cos\left(\dfrac{2\pi}{5}\right) = \dfrac{-1 + \sqrt{5}}{2}$, et

$$\boxed{\cos\left(\frac{2\pi}{5}\right) = \frac{-1 + \sqrt{5}}{4}}\ ①$$