# TD D'OPTION INFORMATIQUE 1
# Déduction naturelle

## Corrigé du TD

## 1 Premières preuves

1.

$$\dfrac{\dfrac{}{\Gamma = p, p \to q \vdash p}\,(\mathrm{Ax}) \qquad \dfrac{}{\Gamma \vdash p \to q}\,(\mathrm{Ax})}{p, p \to q \vdash q}\,(\to_e)$$

2.

$$\dfrac{\dfrac{}{\Gamma \vdash p \lor q}\,(\mathrm{Ax}) \qquad \dfrac{\dfrac{}{\Gamma, p \vdash p}\,(\mathrm{Ax})}{\Gamma, p \vdash q \lor p}\,(\lor_i^d) \qquad \dfrac{\dfrac{}{\Gamma, q \vdash q}\,(\mathrm{Ax})}{\Gamma, q \vdash q \lor p}\,(\lor_i^g)}{\Gamma = p \lor q \vdash q \lor p}\,(\lor_e)$$

3.

$$\dfrac{\dfrac{\dfrac{}{\Gamma \vdash q \to r}\,(\mathrm{Ax}) \qquad \dfrac{\dfrac{}{\Gamma \vdash p \to q}\,(\mathrm{Ax}) \qquad \dfrac{}{\Gamma \vdash p}\,(\mathrm{Ax})}{\Gamma \vdash q}\,(\to_e)}{\Gamma = p \to q, q \to r, p \vdash r}\,(\to_e)}{p \to q, q \to r \vdash p \to r}\,(\to_i)$$

4.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash p \land \neg q}\,(\mathrm{Ax})}{\Gamma \vdash \neg q}\,(\land_e^d) \qquad \dfrac{\dfrac{}{\Gamma \vdash p \to q}\,(\mathrm{Ax}) \qquad \dfrac{\dfrac{}{\Gamma \vdash p \land \neg q}\,(\mathrm{Ax})}{\Gamma \vdash p}\,(\land_e^g)}{\Gamma \vdash q}\,(\to_e)}{\Gamma = p \to q,\ p \land \neg q \vdash \bot}\,(\neg_e)}{p \to q \vdash \neg(p \land \neg q)}\,(\neg_i)}{\vdash (p \to q) \to \neg(p \land \neg q)}\,(\to_i)$$

5.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash \neg p \to p}\,(\mathrm{Ax}) \qquad \dfrac{}{\Gamma \vdash \neg p}\,(\mathrm{Ax})}{\Gamma \vdash p}\,(\to_e) \qquad \dfrac{}{\Gamma \vdash \neg p}\,(\mathrm{Ax})}{\Gamma = \neg p \to p,\ \neg p \vdash \bot}\,(\neg_e)}{\neg p \to p \vdash p}\,(\mathrm{Abs})$$

6.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash \neg(p \land \neg q)}\,(\mathrm{Ax}) \qquad \dfrac{\dfrac{}{\Gamma \vdash p}\,(\mathrm{Ax}) \qquad \dfrac{}{\Gamma \vdash \neg q}\,(\mathrm{Ax})}{\Gamma \vdash p \land \neg q}\,(\land_i)}{\Gamma = \neg(p \land \neg q),\ p,\ \neg q \vdash \bot}\,(\neg_e)}{\neg(p \land \neg q),\ p \vdash q}\,(\mathrm{Abs})}{\neg(p \land \neg q) \vdash p \to q}\,(\to_i)}{\vdash \neg(p \land \neg q) \to (p \to q)}\,(\to_i)$$

# 2 Lois de Morgan

1.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{(Ax)}
\UnaryInfC{$\Gamma \vdash \neg(p \vee q)$}
\AxiomC{}
\RightLabel{(Ax)}
\UnaryInfC{$\Gamma, p \vdash p$}
\RightLabel{($\vee_i^g$)}
\UnaryInfC{$\Gamma, p \vdash p \vee q$}
\RightLabel{($\neg_e$)}
\BinaryInfC{$\Gamma, p \vdash \bot$}
\RightLabel{($\neg_i$)}
\UnaryInfC{$\Gamma \vdash \neg p$}
\AxiomC{}
\RightLabel{(Ax)}
\UnaryInfC{$\Gamma \vdash \neg(p \vee q)$}
\AxiomC{}
\RightLabel{(Ax)}
\UnaryInfC{$\Gamma, q \vdash q$}
\RightLabel{($\vee_i^d$)}
\UnaryInfC{$\Gamma, q \vdash p \vee q$}
\RightLabel{($\neg_e$)}
\BinaryInfC{$\Gamma, q \vdash \bot$}
\RightLabel{($\neg_i$)}
\UnaryInfC{$\Gamma \vdash \neg q$}
\RightLabel{($\wedge_i$)}
\BinaryInfC{$\Gamma = \neg(p \vee q) \vdash \neg p \wedge \neg q$}
\end{prooftree}
$$

2.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{(Ax)}
\UnaryInfC{$\Gamma \vdash \neg p \vee \neg q$}
\AxiomC{}
\RightLabel{(Ax)}
\UnaryInfC{$\Gamma, \neg p \vdash \neg p$}
\AxiomC{}
\RightLabel{(Ax)}
\UnaryInfC{$\Gamma, \neg p \vdash p \wedge q$}
\RightLabel{($\wedge_e^g$)}
\UnaryInfC{$\Gamma, \neg p \vdash p$}
\RightLabel{($\neg_e$)}
\BinaryInfC{$\Gamma, \neg p \vdash \bot$}
\AxiomC{}
\RightLabel{(Ax)}
\UnaryInfC{$\Gamma, \neg q \vdash \neg q$}
\AxiomC{}
\RightLabel{(Ax)}
\UnaryInfC{$\Gamma, \neg q \vdash p \wedge q$}
\RightLabel{($\wedge_e^d$)}
\UnaryInfC{$\Gamma, \neg q \vdash q$}
\RightLabel{($\neg_e$)}
\BinaryInfC{$\Gamma, \neg q \vdash \bot$}
\RightLabel{($\vee_e$)}
\TrinaryInfC{$\Gamma = \neg p \vee \neg q,\, p \wedge q \vdash \bot$}
\RightLabel{($\neg_i$)}
\UnaryInfC{$\neg p \vee \neg q \vdash \neg(p \wedge q)$}
\end{prooftree}
$$

3.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{(Ax)}
\UnaryInfC{$\Gamma \vdash p \vee q$}
\AxiomC{}
\RightLabel{(Ax)}
\UnaryInfC{$\Gamma,\, p \vdash \neg p \wedge \neg q$}
\RightLabel{($\wedge_e^g$)}
\UnaryInfC{$\Gamma,\, p \vdash \neg p$}
\AxiomC{}
\RightLabel{(Ax)}
\UnaryInfC{$\Gamma,\, p \vdash p$}
\RightLabel{($\neg_e$)}
\BinaryInfC{$\Gamma,\, p \vdash \bot$}
\AxiomC{}
\RightLabel{(Ax)}
\UnaryInfC{$\Gamma,\, q \vdash \neg p \wedge \neg q$}
\RightLabel{($\wedge_e^d$)}
\UnaryInfC{$\Gamma,\, q \vdash \neg q$}
\AxiomC{}
\RightLabel{(Ax)}
\UnaryInfC{$\Gamma,\, q \vdash q$}
\RightLabel{($\neg_e$)}
\BinaryInfC{$\Gamma,\, q \vdash \bot$}
\RightLabel{($\vee_e$)}
\TrinaryInfC{$\Gamma = \neg p \wedge \neg q,\, p \vee q \vdash \bot$}
\RightLabel{($\neg_i$)}
\UnaryInfC{$\neg p \wedge \neg q \vdash \neg(p \vee q)$}
\end{prooftree}
$$

4.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{(TE)}
\UnaryInfC{$\vdash p \vee \neg p$}
\AxiomC{}
\RightLabel{(Ax)}
\UnaryInfC{$\Gamma \vdash \neg(p \wedge q)$}
\AxiomC{}
\RightLabel{(Ax)}
\UnaryInfC{$\Gamma \vdash p$}
\AxiomC{}
\RightLabel{(Ax)}
\UnaryInfC{$\Gamma \vdash q$}
\RightLabel{($\wedge_i$)}
\BinaryInfC{$\Gamma \vdash p \wedge q$}
\RightLabel{($\neg_e$)}
\BinaryInfC{$\Gamma = \neg(p \wedge q),\, p,\, q \vdash \bot$}
\RightLabel{($\neg_i$)}
\UnaryInfC{$\neg(p \wedge q),\, p \vdash \neg q$}
\RightLabel{($\vee_i^d$)}
\UnaryInfC{$\neg(p \wedge q),\, p \vdash \neg p \vee \neg q$}
\AxiomC{}
\RightLabel{(Ax)}
\UnaryInfC{$\neg(p \wedge q),\, \neg p \vdash \neg p$}
\RightLabel{($\vee_i^g$)}
\UnaryInfC{$\neg(p \wedge q),\, \neg p \vdash \neg p \vee \neg q$}
\RightLabel{($\vee_e$)}
\TrinaryInfC{$\neg(p \wedge q) \vdash \neg p \vee \neg q$}
\end{prooftree}
$$

# 3 Equivalence des règles de la logique classique

1.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{(TE)}
\UnaryInfC{$\Gamma \vdash \varphi \vee \neg\varphi$}
\AxiomC{}
\RightLabel{(Ax)}
\UnaryInfC{$\Gamma, \varphi \vdash \varphi$}
\AxiomC{$\Gamma, \neg\varphi \vdash \bot$}
\RightLabel{($\bot_e$)}
\UnaryInfC{$\Gamma, \neg\varphi \vdash \varphi$}
\RightLabel{($\vee_e$)}
\TrinaryInfC{$\Gamma \vdash \varphi$}
\end{prooftree}
$$

2.

$$
\begin{prooftree}
\AxiomC{$\Gamma, \neg\varphi \vdash \bot$}
\RightLabel{($\neg_i$)}
\UnaryInfC{$\Gamma \vdash \neg\neg\varphi$}
\RightLabel{($\neg\neg_e$)}
\UnaryInfC{$\Gamma \vdash \varphi$}
\end{prooftree}
$$

3.

$$
\begin{prooftree}
\AxiomC{$\Gamma \vdash \neg\neg\varphi$}
\RightLabel{(Aff)}
\UnaryInfC{$\Gamma, \neg\varphi \vdash \neg\neg\varphi$}
\AxiomC{}
\RightLabel{(Ax)}
\UnaryInfC{$\Gamma, \neg\varphi \vdash \neg\varphi$}
\RightLabel{($\neg_e$)}
\BinaryInfC{$\Gamma, \neg\varphi \vdash \bot$}
\RightLabel{(Abs)}
\UnaryInfC{$\Gamma \vdash \varphi$}
\end{prooftree}
$$

4.
$$\dfrac{\dfrac{}{\Gamma \vdash \varphi \vee \neg\varphi}\,(\text{TE}) \qquad \dfrac{}{\Gamma, \varphi \vdash \varphi}\,(\text{Ax}) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\Gamma \vdash \neg\neg\varphi}{\Gamma, \neg\varphi \vdash \neg\neg\varphi}\,(\text{Aff}) \qquad \dfrac{}{\Gamma, \neg\varphi \vdash \neg\varphi}\,(\text{Ax})}{\Gamma, \neg\varphi \vdash \bot}\,(\neg_e)}{\Gamma, \neg\varphi \vdash \varphi}\,(\bot_e)}{}}{\Gamma \vdash \varphi}\,(\vee_e)$$

5.
$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma' \vdash \varphi}\,(Ax)}{\Gamma' \vdash \varphi \vee \neg\varphi}\,(\vee_i^g) \qquad \dfrac{}{\Gamma' \vdash \neg(\varphi \vee \neg\varphi)}\,(Ax)}{\Gamma' = \Gamma, \neg(\varphi \vee \neg\varphi), \varphi \vdash \bot}\,(\neg_e)}{\Gamma, \neg(\varphi \vee \neg\varphi) \vdash \neg\varphi}\,(\neg_i)}{\Gamma, \neg(\varphi \vee \neg\varphi) \vdash \varphi \vee \neg\varphi}\,(\vee_i^d) \qquad \dfrac{}{\Gamma, \neg(\varphi \vee \neg\varphi) \vdash \neg(\varphi \vee \neg\varphi)}\,(Ax)}{\dfrac{\Gamma, \neg(\varphi \vee \neg\varphi) \vdash \bot}{\Gamma \vdash \varphi \vee \neg\varphi}\,(\text{Abs})}\,(\neg_e)$$

6. Il suffit de combiner les arbres de la 5 et de la 2 (dans laquelle on remplace $\varphi$ par $\varphi \vee \neg\varphi$).

# 4 Formules usuelles

1.
$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash (A \wedge B) \to C}\,(Ax) \qquad \dfrac{\dfrac{}{\Gamma \vdash A}\,(Ax) \quad \dfrac{}{\Gamma \vdash B}\,(Ax)}{\Gamma \vdash A \wedge B}\,(\wedge_i)}{\Gamma = (A \wedge B) \to C,\ A,\ B \vdash C}\,(\to_e)}{(A \wedge B) \to C, A \vdash B \to C}\,(\to_i)}{(A \wedge B) \to C \vdash A \to (B \to C)}\,(\to_i)}{\vdash ((A \wedge B) \to C) \to (A \to (B \to C))}\,(\to_i)$$

2.
$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash \neg B}\,(Ax) \qquad \dfrac{\dfrac{}{\Gamma \vdash A \to B}\,(Ax) \quad \dfrac{}{\Gamma \vdash A}\,(Ax)}{\Gamma \vdash B}\,(\to_e)}{\Gamma = A \to B,\ \neg B,\ A \vdash \bot}\,(\neg_e)}{A \to B, \neg B \vdash \neg A}\,(\neg_i)}{A \to B \vdash \neg B \to \neg A}\,(\to_i)}{\vdash (A \to B) \to (\neg B \to \neg A)}\,(\to_i)$$

3.
$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash \neg B \to \neg A}\,(Ax) \quad \dfrac{}{\Gamma \vdash \neg B}\,(Ax)}{\Gamma \vdash \neg A}\,(\to_e) \qquad \dfrac{}{\Gamma \vdash A}\,(Ax)}{\Gamma = \neg B \to \neg A,\ A,\ \neg B \vdash \bot}\,(\neg_e)}{\neg B \to \neg A, A \vdash B}\,(\text{Abs})}{\neg B \to \neg A \vdash A \to B}\,(\to_i)}{\vdash (\neg B \to \neg A) \to (A \to B)}\,(\to_i)$$

4.
$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash \neg\neg\neg A}\,(Ax) \qquad \dfrac{\dfrac{\dfrac{}{\Gamma, \neg A \vdash A}\,(Ax) \quad \dfrac{}{\Gamma, \neg A \vdash \neg A}\,(Ax)}{\Gamma, \neg A \vdash \bot}\,(\neg_e)}{\Gamma \vdash \neg\neg A}\,(\neg_i)}{\Gamma = \neg\neg\neg A,\ A \vdash \bot}\,(\neg_e)}{\neg\neg\neg A \vdash \neg A}\,(\neg_i)}{\vdash \neg\neg\neg A \to \neg A}\,(\to_i)$$

5.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{$(Ax)$}
\UnaryInfC{$\Gamma \vdash (A \to C) \lor (B \to C)$}
\AxiomC{}
\RightLabel{$(Ax)$}
\UnaryInfC{$\Gamma,\ A \to C \vdash A \to C$}
\AxiomC{}
\RightLabel{$(Ax)$}
\UnaryInfC{$\Gamma,\ A \to C \vdash A \land B$}
\RightLabel{$(\land_e)$}
\UnaryInfC{$\Gamma,\ A \to C \vdash A$}
\RightLabel{$(\to_e)$}
\BinaryInfC{$\Gamma,\ A \to C \vdash C$}
\AxiomC{Symétrique}
\RightLabel{$(\to_e$}
\UnaryInfC{$\Gamma,\ B \to C \vdash C$}
\RightLabel{$(\lor_e)$}
\TrinaryInfC{$\Gamma = (A \to C) \lor (B \to C),\ A \land B \vdash C$}
\RightLabel{$(\to_i)$}
\UnaryInfC{$(A \to C) \lor (B \to C) \vdash (A \land B) \to C$}
\RightLabel{$(\to_i)$}
\UnaryInfC{$\vdash ((A \to C) \lor (B \to C)) \to ((A \land B) \to C)$}
\end{prooftree}
$$

6. Utiliser un tiers-exclu sur A et non A
   (suite non corrigée)