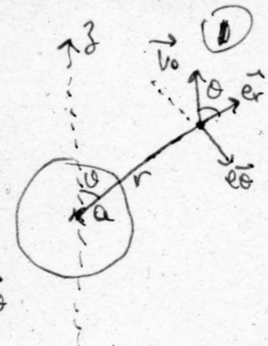
1



· Formule : $\boxed{\vec{\mathrm{rot}}\,\vec{A} = \frac{1}{r\sin\theta}\left(\frac{\partial A_\varphi \sin\theta}{\partial\theta} - \frac{\partial A_\theta}{\partial\varphi}\right)\vec{e_r} + \left(\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\varphi} - \frac{1}{r}\frac{\partial A_\varphi r}{\partial r}\right)\vec{e_\theta} + \frac{1}{r}\left(\frac{\partial A_\theta r}{\partial r} - \frac{\partial A_r}{\partial\theta}\right)\vec{e_\varphi}}$

· Conditions limites · $\boxed{\lim_{r\to\infty} v_r = V_0\cos\theta \quad \lim_{r\to\infty} v_\theta = -V_0\sin\theta}$ $\boxed{\lim_{r\to a} v_r = 0}$ $\boxed{\lim_{r\to a} v_\theta = 0}$ car $\delta F^{visc}_{couche/sphère} = \eta S \frac{\partial v_\theta}{\partial r}\vec{e_\theta}$

FORME de $\vec{v}$:

· Fluide incompressible $\Rightarrow \mathrm{div}(\rho\vec{v}) + \frac{\partial\rho}{\partial t} = 0 \Rightarrow \mathrm{div}\,\vec{v} = 0 \Rightarrow \exists\,\vec{A} \,/\, \boxed{\vec{v} = \vec{\mathrm{rot}}\,\vec{A}}$

$\vec{v}$: { − pas de dépendance en φ ; − pas de composante selon φ } $\Rightarrow \boxed{v_r = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\varphi \sin\theta)}$ $\boxed{v_\theta = -\frac{1}{r}\frac{\partial A_\varphi r}{\partial r}}$

$\Rightarrow$ Il faut rechercher $\boxed{A_\varphi = f(r)\,g(\theta)}$.

$\Rightarrow \boxed{v_r = \frac{f}{r\sin\theta}\frac{d}{d\theta}(g\sin\theta)}$ $\boxed{v_\theta = -\frac{1}{r}\,g\,\frac{d}{dr}(rf)}$

· Conditions limites : $\lim_{r\to\infty}\left[-\frac{1}{r}\frac{d}{dr}(rf)\right]\times g(\theta) = -V_0\sin\theta$

$\lim_{r\to\infty}\left[\frac{f}{r}\right]\times\frac{1}{\sin\theta}\frac{d}{d\theta}(g\sin\theta) = V_0\cos\theta$

$\Rightarrow$ on pose $\boxed{g(\theta) = \sin\theta}$

$\Rightarrow \boxed{v_r = \frac{2f}{r}\times\cos\theta \quad v_\theta = -\frac{1}{r}\frac{d}{dr}(rf)\times\sin\theta}$

ÉQUATION de NAVIER-STOCKES :

$$\mu\left[\frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\,\vec{v}\right] = -\vec{\mathrm{grad}}\,p + \mu\vec{g} + \eta\,\Delta\vec{v}.$$

− Régime stationnaire $\Rightarrow \frac{\partial\vec{v}}{\partial t}$.

· $\vec{v} = \vec{\mathrm{rot}}\,\vec{A} \Rightarrow \vec{v}\cdot\vec{\nabla} = (\vec{\nabla}\wedge\vec{A})\cdot\vec{\nabla} = \vec{0} \Rightarrow \vec{0} = -\vec{\mathrm{grad}}\,p + \eta\,\Delta\vec{v}$.

· On néglige le poids $\Rightarrow \mu\vec{g} \simeq 0$

· $\vec{\mathrm{rot}}(\vec{\mathrm{rot}}\,\vec{v}) = \vec{\mathrm{grad}}(\underbrace{\mathrm{div}\,\vec{v}}_{=0}) - \Delta\vec{v} \Rightarrow \vec{0} = -\vec{\mathrm{grad}}\,p - \eta\,\vec{\mathrm{rot}}(\vec{\mathrm{rot}}\,\vec{v})$ 1

$\Rightarrow \vec{0} = -\vec{\mathrm{rot}}(\vec{\mathrm{grad}}\,p) - \eta\,\vec{\mathrm{rot}}\left[\vec{\mathrm{rot}}(\vec{\mathrm{rot}}\,\vec{v})\right]$

$\Rightarrow \boxed{\vec{\mathrm{rot}}\underbrace{\left[\vec{\mathrm{rot}}(\underbrace{\vec{\mathrm{rot}}\,\vec{v}}_{\vec{K}})\right]}_{\vec{W}} = \vec{0}}$

RESOLUTION :

· $\vec{K} = \vec{\mathrm{rot}}\,\vec{v} \Rightarrow K_\varphi = \frac{1}{r}\frac{\partial}{\partial r}(r v_\theta) - \frac{1}{r}\frac{\partial v_r}{\partial\theta}$

$= \frac{1}{r}\frac{\partial}{\partial r}\left[-\sin\theta\frac{d}{dr}(rf)\right] - \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{2\cos\theta f}{r}\right)$

$= \frac{1}{r}\left[-\sin\theta\frac{d^2}{dr^2}(rf) + 2\sin\theta\frac{f}{r}\right] \Rightarrow \boxed{K_\varphi = \left[\frac{2f}{r^2} - \frac{1}{r}\frac{d^2}{dr^2}(rf)\right]\sin\theta}$

· $\vec{W} = \vec{\mathrm{rot}}\,\vec{K} \Rightarrow W_r = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left[\sin^2\theta\left(\frac{2f}{r^2} - \frac{1}{r}\frac{d^2}{dr^2}(rf)\right)\right] \Rightarrow \boxed{W_r = 2\cos\theta\left[\frac{2f}{r^3} - \frac{1}{r^2}\frac{d^2}{dr^2}(rf)\right]}$

$W_\theta = -\frac{1}{r}\frac{\partial}{\partial r}\left[\frac{2f}{r} - \frac{d^2}{dr^2}(rf)\right]\sin\theta \Rightarrow \boxed{W_\theta = \sin\theta\left[\frac{1}{r}\frac{d^3}{dr^3}(rf) - \frac{1}{r}\frac{d}{dr}\left(\frac{2f}{r}\right)\right]}$

· $\vec{\mathrm{rot}}\,\vec{W} = \vec{0} \Rightarrow \frac{\partial}{\partial r}(r W_\theta) = \frac{\partial W_r}{\partial\theta}$

$\Rightarrow \boxed{\frac{d^4}{dr^4}(rf) - \frac{d^2}{dr^2}\left(\frac{2f}{r}\right) = -2\left[\frac{2f}{r^3} - \frac{1}{r^2}\frac{d^2}{dr^2}(rf)\right]}$

RESOLUTION de l'équation en $f(r)$:

$$\boxed{\frac{d^4}{dr^4}(rf) - \frac{d^2}{dr^2}\left(\frac{2f}{r}\right) = -2\left[\frac{2f}{r^3} - \frac{1}{r^2}\frac{d^2}{dr^2}(rf)\right]}$$

rappel :
- $v_r = \frac{2f}{r} \times \cos\theta$
- $v_\theta = -\frac{1}{r}\frac{d}{dr}(rf) \times \sin\theta$
- $\lim_{r\to\infty}\left[\frac{1}{r}\frac{d}{dr}(rf)\right] = V_0$
- $\lim_{r\to\infty}\left[\frac{f}{r}\right] = V_0/2$
- $\lim_{r\to a}\left(\frac{2f}{r}\right) = 0 \quad (\alpha)$
- $\lim_{r\to a}\left[\frac{1}{r}\frac{d}{dr}(rf)\right] = 0 \quad (\beta)$

On cherche $f(r) = \sum_{n=-\infty}^{+\infty} \alpha_n r^n$.

- $\frac{d^4}{dr^4}(rf) = \sum_n \alpha_n (n+1) n (n-1)(n-2) r^{n-3}$
- $\frac{d^2}{dr^2}\left(\frac{f}{r}\right) = \sum_n \alpha_n (n-1)(n-2) r^{n-3}$
- $\frac{f}{r^3} = \sum_n \alpha_n r^{n-3}$
- $\frac{1}{r^2}\frac{d^2}{dr^2}(rf) = \sum_n \alpha_n (n+1) n r^{n-3}$.

$\Rightarrow (n+1)n(n-1)(n-2) - 2(n-1)(n-2) = -2 \times [2 - n(n+1)]$

$\Rightarrow (n-2)(n-1)\underbrace{(-2+n^2+n)}_{(n+2)(n-1)} = -4 + 2n(n+1)$.

$\Rightarrow (n-2)(n-1)^2(n+2) = -4 + 2n^2 + 2n = 2(n-1)(n+2)$

$\Rightarrow (n-1)(n+2)(n-2)(n-1) - 2(n-1)(n+2) = 0 \Rightarrow (n-1)(n+2)\overbrace{[(n-2)(n-1) - 2]}^{n^2-3n} = 0$

$\Rightarrow (n-1)(n+2)n(n-3) = 0 \Rightarrow \boxed{f(r) = Ar + \frac{B}{r^2} + C + Dr^3} \Rightarrow \boxed{\frac{f}{r} = A + \frac{B}{r^3} + \frac{C}{r} + Dr^2}$

$\Rightarrow \boxed{\frac{1}{r}\frac{d}{dr}(rf) = 2A - \frac{B}{r^3} + \frac{C}{r} + 4Dr^2}$

- $\lim_{r\to\infty}\left(\frac{f}{r}\right) = \frac{V_0}{2} \neq \infty \Rightarrow \boxed{D=0}$ et $\boxed{A = V_0/2}$

$(\alpha)(\beta) \Rightarrow \begin{cases} \frac{V_0}{2} + \frac{B}{a^3} + \frac{C}{a} = 0 \\ V_0 - \frac{B}{a^3} + \frac{C}{a} = 0 \end{cases} \Rightarrow \boxed{C = -\frac{3V_0 a}{4}}$ et $\boxed{B = \frac{V_0 a^3}{4}} \Rightarrow f(r) = \frac{V_0 r}{2} + \frac{V_0 a^3}{4r^2} - \frac{3V_0 a}{4}$

$\Rightarrow \boxed{f(r) = \frac{V_0}{4}\left(-3a + 2r + \frac{a^3}{r^2}\right)}$

- $\boxed{v_r = \cancel{V_0}\, \frac{2f}{r}\cos\theta = V_0\left(1 - \frac{3a}{2r} + \frac{a^3}{2r^3}\right)\cos\theta}$ et $v_\theta = -\frac{1}{r}\frac{d}{dr}(rf)\sin\theta \Rightarrow \boxed{v_\theta = -V_0 \times \left(1 - \frac{3a}{4r} - \frac{a^3}{4r^3}\right)\sin\theta}$

FORCE DE PRESSION :

Equation (1) $\Rightarrow \overrightarrow{\text{grad}}\, p = -\eta\, \overrightarrow{\text{rot}}(\overrightarrow{\text{rot}}\, \vec{v}) = -\eta\, \vec{W} \Rightarrow \frac{\partial p}{\partial r} = -\eta W_r = -\eta \times 2\cos\theta \underbrace{\left[\frac{2f}{r^3} - \frac{1}{r^2}\frac{d^2}{dr^2}(rf)\right]}_{F}$

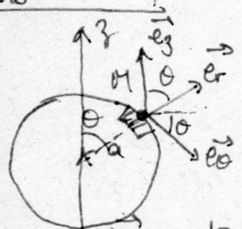
- $rf = \frac{V_0}{4}\left(-3ar + 2r^2 + \frac{a^3}{r}\right) \Rightarrow \frac{d}{dr}(rf) = \frac{V_0}{4}\left(-3a + 4r - \frac{a^3}{r^2}\right) \Rightarrow \frac{d^2}{dr^2}(rf) = \frac{V_0}{4}\left(4 + \frac{2a^3}{r^3}\right)$

$\Rightarrow F = \frac{V_0}{4}\left(-\frac{6a}{r^3} + \cancel{\frac{4}{r^2}} + \cancel{\frac{2a^3}{r^5}}\right) - \frac{V_0}{4}\left(\cancel{\frac{4}{r^2}} + \cancel{\frac{2a^3}{r^5}}\right) = \frac{V_0}{4} \times -\frac{6a}{r^3}$

$\Rightarrow \frac{\partial p}{\partial r} = \frac{2\eta\cos\theta}{4} \times \frac{6aV_0}{r^3} \Rightarrow \boxed{p = P_0 - \frac{3aV_0}{2r^2}\eta\cos\theta}$ car $\boxed{\lim_{r\to\infty} p = P_0}$

$$F_{p_z} = -\int_{\varphi=0}^{2\pi}\int_{\theta=0}^{\pi}\left(P_0 - \frac{3\eta V_0}{2a}\cos\theta\right)a^2\sin\theta\, d\theta\, \underbrace{\vec{e_r}\cdot\vec{e_z}}_{\cos\theta} \rightarrow \frac{2}{3}$$

$$= 2\pi\frac{3\eta V_0}{2a}a^2\int_0^{\pi}\cos^2\theta\sin\theta\, d\theta = \frac{3\eta V_0 a}{2}\left[-\frac{\cos^3\theta}{3}\right]_0^{\pi} \times 2\pi \Rightarrow \boxed{F_{p_z} = 2\pi\eta V_0 a}$$



$d\vec{S}_M = +dS_M\, \vec{e_r}$

$dS_M = a^2\sin\theta\, d\theta\, d\varphi$

FORCE DE VISCOSITÉ

$d\vec{F}_{\text{viscosité}} = d\vec{F}_{\text{sphère/eau}} = -\eta\, dS\, \frac{\partial v_\theta}{\partial r}\vec{e_\theta} \Rightarrow d\vec{F}_{\text{eau/sphère}} = \eta\, dS \times \left.\frac{\partial v_\theta}{\partial r}\right|_{r=a}\vec{e_\theta}$

$\frac{\partial v_\theta}{\partial r} = -V_0\sin\theta\left(+\frac{3a}{4r^2} + \frac{3a^3}{4r^4}\right) \Rightarrow d\vec{F}_{\text{eau/sphère}} = -\eta\, dS \times V_0\sin\theta \times \frac{3}{2a}\vec{e_\theta}$

$\Rightarrow \vec{e_z}\cdot\vec{F}_{\text{visc}} = \int_{\varphi=0}^{2\pi}d\varphi\int_{\theta=0}^{\pi} -\eta V_0\sin\theta\frac{3}{2a} \times a^2\sin\theta\, d\theta \times \underbrace{\vec{e_\theta}\cdot\vec{e_z}}_{-\sin\theta} \Rightarrow F_{\text{visc},z} = 2\pi\eta V_0 \cancel{\sin\theta} \times \frac{3a}{2} \times \int_{\theta=0}^{\pi}\cancel{\sin}(1-\cos^2\theta)\sin\theta\, d\theta$

$= \pi\eta V_0 \cancel{\sin\theta} \times 3a \times \underbrace{\left[-\cos\theta + \frac{\cos^3\theta}{3}\right]_0^{\pi}}_{2\times\frac{2}{3}}$

$\Rightarrow \boxed{F_{\text{visc},z} = 4\pi\eta V_0 a}$

$\Rightarrow \boxed{\vec{F} = 6\pi\eta a V_0 \vec{u_z}}$