Q9.2 $(E) \Leftrightarrow X'(t) = \underbrace{\begin{pmatrix} 0 & 1 \\ 2\frac{1-t^2}{(1+t^2)^2} & \frac{2t}{1+t^2} \end{pmatrix}}_{A_t} X(t)$

$$\operatorname{tr} A_t = \frac{2t}{1+t^2} \qquad \int \operatorname{tr} A_t \, dt = \ln(1+t^2)$$

$$\Rightarrow W(t) \propto (1+t^2).$$

$$(E) \Leftrightarrow (1+t^2)^2 x''(t) - 2t(1+t^2)x'(t) + 2(t^2-1)x(t) = 0$$

$$\Leftrightarrow (1+2t^2+t^4)\, x''(t) - (2t+2t^3)\, x'(t) + (2t^2-2)\, x(t) = 0.$$

Sol. poly de degré d, unitaire : $x(t) = t^d + \dots$

$$x'(t) = d\,t^{d-1} + \dots$$
$$x''(t) = d(d-1)\,t^{d-2} + \dots$$

$$\Rightarrow (2t^2-2)x(t) = 2\,t^{d+2} + \dots$$
$$-(2t+2t^3)x'(t) = -2d\,t^{d+2} + \dots$$
$$(1+t^2)^2 x''(t) = d(d-1)\,t^{d+2} + \dots$$
$$\underbrace{\left(d(d-1) - 2d + 2\right)}_{d^2-3d+2 \,=\, (d-2)(d-1)} t^{d+2} + \dots = 0$$

$K(t) = \operatorname{Arctan} t$

$K'(t) = \frac{1}{1+t^2}$

$K''(t) = \frac{-2t}{(1+t^2)^2}$

$\frac{K''(t)}{K'(t)} = \frac{-2t}{1+t^2}$

$$\Rightarrow x(t) = t + a \longrightarrow -2t(1+t^2) + 2(t^2-1)(t+a)$$
$$= -2t \cancel{-2t^3} \cancel{+2t^3} + \underbrace{2at^2}_{=0} - 2t - 2a$$
$$= -4t \quad \underline{\underline{\text{non}}}.$$

$$\Rightarrow x(t) = t^2 + at + b$$
$$\hookrightarrow 2(1+t^2)^2 - 2(t+t^3)(2t+a) + 2(t^2-1)(t^2+at+b)$$
$$= 2 + 4t^2 + 2t^4$$
$$-2at - 4t^2 - 2at^3 - 4t^4$$
$$-2b - 2at + 2(b-1)t^2 + 2at^3 + 2t^4$$
$$= 2(1-b) - 4at + 2(b-1)t^2$$

$\begin{cases} a = 0 \\ b = 1 \end{cases} \longrightarrow x(t) = (1+t^2)$

VC $\quad x(t) = K(t)(1+t^2)$

$x'(t) = K'(t)\cdot(1+t^2) + \dots$

$x''(t) = K''(t)\cdot(1+t^2) + 4K'(t)\cdot t + \dots$

$(1+t^2)^{\cancel{3}\,2} K''(t) + 4t\cancel{(1+t^2)}^{\cancel{2}\,1} K'(t) - 2t(1+t^2)K'(t)\cancel{(1+t^2)} = 0$

$(1+t^2)^{\cancel{2}} K''(t) + 2t(\cancel{t^2+1})K'(t) = 0. \quad \frac{K''}{K'} = \frac{-2t}{1+t^2} \quad K' \propto \frac{1}{1+t^2}$

$$x = \sum_{n=0}^{+\infty} a_n t^n \qquad x' = \sum_{n=1}^{+\infty} n a_n t^{n-1} \qquad x'' = \sum_{n=2}^{+\infty} n(n-1) a_n t^{n-2}$$

$$(1+t^2)^2 x'' = (1+2t^2+t^4)\, x''$$

$$= \underbrace{\sum_{n=2}^{+\infty} n(n-1)a_n t^{n-2}}_{\sum_{n=0}^{+\infty} (n+2)(n+1)a_{n+2} t^n} + \underbrace{\sum_{n=2}^{+\infty} 2n(n-1)a_n t^n}_{\sum_{n=0}^{+\infty}} + \underbrace{\sum_{n=2}^{+\infty} n(n-1)a_n t^{n+2}}_{\sum_{n=4}^{+\infty} (n-2)(n-3)a_{n-2} t^n}$$

$$-2t(1+t^2)x' = (-2t-2t^3)x'$$

$$= \sum_{n=1}^{+\infty} -2na_n t^n + \sum_{n=1}^{+\infty} -2na_n t^{n+2}$$

$$= \sum_{n=1}^{+\infty} -2na_n t^n + \underbrace{\sum_{n=3}^{+\infty} -2(n-2)a_{n-2} t^n}_{\sum_{n=2}^{+\infty}}$$

$$2(t^2-1)x = (2t^2-2)x$$

$$= \sum_{n=0}^{+\infty} 2a_n t^{n+2} + \sum_{n=0}^{+\infty} -2a_n t^n \qquad a_3 = \frac{2}{3} a_1$$

$$= \sum_{n=2}^{+\infty} 2a_{n-2} t^n + \sum_{n=0}^{+\infty} -2a_n t^n \qquad a_5 = \frac{-1}{5} a_3$$

Bref : $n=0$: $(2a_2 - 2a_0) = 2(a_2 - a_0)$

$n=1$: $(6a_3 - 2a_1 - 2a_1) = 2(3a_3 - 2a_1)$

$n=2$: $(12a_4 + 4a_2 - 4a_2 + 2a_0 - 2a_2) = 2(6a_4 - a_2 + a_0)$

$n=3$: $20a_5 + 12a_3 - 6a_3 - 2a_1 + 2a_1 - 2a_3$

$$= 2\left[10a_5 + 2a_3\right] = 4\left[5a_5 + a_3\right]$$

$n \geq 4$: $a_{n+2}(n^2+3n+2)$

$+ a_n(2n^2 - 2n - 2n - 2)$

$+ a_{n-2}(n^2 - 5n + 6 - 2n + 4 + 2)$

$$= (n^2+3n+2)a_{n+2} + 2(n^2-2n-1)a_n + (n^2-7n+12)a_{n-2}$$

En particulier ($n=4$) : $30a_6 + 14\underbrace{a_4}_{=0} + 0a_2 = 0$.

$\Rightarrow a_6 = a_4 = 0$

et par réc^{ce} $\forall p \geq 2$, $a_{2p} = 0$.

$\Rightarrow$ on a retrouvé $a_0 \cdot (1+t^2)$.

Mieux $a_{n+2}(n+1)(n+2) + 2\underbrace{(n^2-2n-1)}_{(n-1)^2-2}a_n + (n-4)(n-3)a_{n-2} = 0$

$$(1+t^2)\,\mathrm{Arctan}\,t = (1+t^2)\sum_{p=0}^{+\infty}\frac{(-1)^p}{2p+1}t^{2p+1}$$

$$t - \frac{t^3}{3} + \frac{t^5}{5} \dots$$
$$+\, t^3$$

$$= \sum_{p=0}^{+\infty}\frac{(-1)^p\, t^{2p+1}}{2p+1} + \sum_{p=0}^{+\infty}\frac{(-1)^p\, t^{2p+3}}{2p+1}$$

$$= \sum_{p=0}^{+\infty}\frac{(-1)^p\, t^{2p+1}}{2p+1} + \sum_{p=1}^{+\infty}\frac{(-1)^{p-1}\, t^{2p+1}}{2p-1}$$

$$= t + \sum_{p=1}^{+\infty}\cancel{\frac{(-1)^p}{2p+1}}\,(-1)^p\left[\frac{1}{2p+1}-\frac{1}{2p-1}\right]t^{2p+1}$$

$$= t + \sum_{p=1}^{+\infty}\frac{2\cdot(-1)^p}{4p^2-1}\,t^{2p+1}$$

$$a_3 = \frac{-2}{3}\ (p=1)$$

$$a_5 = \frac{2}{15} = \frac{-a_3}{5}\ (p=2)$$

$$\Rightarrow \quad a_1 = 1 \ \wedge\ a_{2p+1} = \frac{2\cdot(-1)^p}{4p^2-1}\quad \forall p \geq 1$$

$$(2p+3)(2p+2)\,a_{2p+3} + 2\left[\cancel{(2p+1)^2}\,(2p)^2 - 2\right]a_{2p+1} + (2p-3)(2p-2)\,a_{2p-1} \overset{?}{=} 0$$

$$(2p+3)(p+1)\cdot a_{2p+3} + (4p^2-2)\cdot a_{2p+1} + (2p-3)(p-1)\cdot a_{2p-1} \overset{?}{=} 0$$

$$\frac{-(2p+3)(p+1)}{4(p+1)^2-1} + \frac{(4p^2-2)}{4p^2-1} - \frac{(2p-3)(p-1)}{4(p-1)^2-1}$$

$$= \frac{-(\cancel{2p+3})(p+1)}{(2p+1)(\cancel{2p+3})} + \frac{4p^2-2}{(2p-1)(2p+1)} - \frac{(\cancel{2p-3})(p-1)}{(2p-1)(\cancel{2p-3})}$$

$$= \frac{-(p+1)(2p-1)}{(2p-1)(2p+1)} + \frac{4p^2-2}{(2p-1)(2p+1)} - \frac{(2p+1)(p-1)}{(2p-1)(2p+1)}$$

$$= \frac{-(2p^2+p-1) + 4p^2-2 - (2p^2-p-1)}{\dots}$$

$$= \frac{(4p^2-2p^2-2p^2) + (p-p) + (1-2+1)}{\dots} = 0 \quad \checkmark.$$