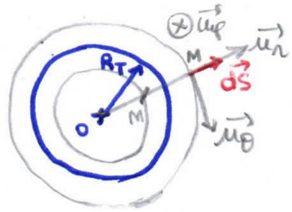
Profil de masse volumique au sein de la Terre

1). th. de Gauss $\displaystyle\oiint_\Sigma \vec{E}\cdot\vec{dS} = \frac{Q_{int}}{\varepsilon_0}$ $\longleftrightarrow$ $\displaystyle\oiint_\Sigma \vec{g}\cdot\vec{dS} = -4\pi\mathcal{G} M_{int}$

2). $\rho(M) = \text{cste} = \dfrac{M_T}{\frac{4}{3}\pi R_T^3}$



distr de masses: $\mathcal{D}_m$

- $\left.\begin{array}{l}(M, \vec{u_r}, \vec{u_\theta}) \\ (M, \vec{u_r}, \vec{u_\varphi})\end{array}\right\}$ plans de sym de $\mathcal{D}_m$ $\Rightarrow \vec{g}(M) \parallel \vec{u_r}$: $\underline{\vec{g}(M) = -g(M)\,\vec{u_r}}$

- $\mathcal{D}_m$ invariante par rotat° autour de O $\Rightarrow \underline{\vec{g}(M) = -g(r)\,\vec{u_r}}$.

- ~~surface~~ de Gauss $(\mathcal{S})$: sphère de centre O de rayon $r = OM$

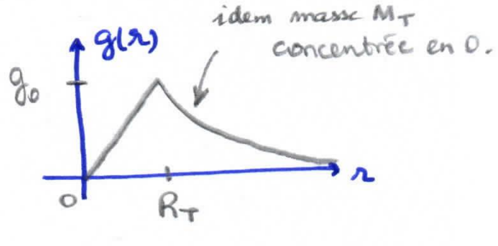
  - $\displaystyle\oiint_{(\mathcal{S})} \vec{g}(M)\cdot\vec{dS} = \oiint_{(\mathcal{S})} -g(r)\,\vec{u_r}\cdot dS\,\vec{u_r} = -g(r)\oiint_{(\mathcal{S})} dS = -g(r)\cdot 4\pi r^2$

  - $M_{int} = \begin{cases} M_T & \text{si } r > R \\ \rho\cdot\frac{4}{3}\pi r^3 & \text{si } r < R \end{cases}$

d'ap. l'analogue du th. de Gauss pour la gravitat°

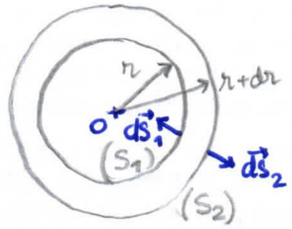
$$-g(r)\cdot\cancel{4\pi} r^2 = \begin{cases} -\cancel{4\pi}\mathcal{G} M_T & \text{si } r > R \\ -\cancel{4\pi}\mathcal{G}\dfrac{M_T}{\cancel{\frac{4}{3}\pi} R_T^3}\cdot\cancel{\frac{4}{3}\pi} r^3 & \text{si } r < R \end{cases}$$

$$\boxed{\vec{g}(M) = \begin{cases} \dfrac{\mathcal{G} M_T}{r^2}\,\vec{u_r} & \text{si } r > R \\ \dfrac{\mathcal{G} M_T\, r}{R_T^3}\,\vec{u_r} & \text{si } r < R \end{cases}}$$



3) $\underline{g_0} = g(r = R_T) = \dfrac{\mathcal{G} M_T}{R_T^2} = \underline{9{,}8\ \text{m}\cdot\text{s}^{-2}}$ ← valeur cohérente

4).



- $\displaystyle\oiint_\Sigma \vec{g}(M)\cdot\vec{dS} = \iint_{S_1} -g(r)\,\vec{u_r}\cdot\overbrace{dS(-\vec{u_r})}^{\vec{dS_1}} + \iint_{S_2} -g(r+dr)\,\vec{u_r}\cdot\overbrace{dS\,\vec{u_r}}^{\vec{dS_2}}$

$= g(r)\,S_1 - g(r+dr)\,S_2$

$= g(r)\cdot 4\pi r^2 - g(r+dr)\cdot 4\pi(r+dr)^2$

$= -4\pi\left[g(r+dr)\cdot(r+dr)^2 - g(r)\cdot r^2\right]$

$\simeq -4\pi\cdot\dfrac{d}{dr}(r^2 g(r))\cdot dr$ DL à l'ordre 1 en dr

- $M_{int} \simeq \underbrace{\rho(r)}_{\simeq\ \text{uniforme sur le volume mésoscopique}}\cdot\ \underbrace{4\pi r^2 dr}$ volume entre les 2 sphères à l'ordre 1 en dr th. de Gauss pour la gravitat° :

$$+\cancel{4\pi}\frac{d}{dr}(r^2 g(r))\cancel{dr} = +\cancel{4\pi}\mathcal{G}\,\rho(r)\cdot 4\pi r^2\cancel{dr}$$

$$\Leftrightarrow \boxed{\rho(r) = \frac{\frac{d}{dr}(r^2 g(r))}{4\pi r^2 \mathcal{G}}} \qquad (*)$$

pour $r \in [0, R_N[$ $\quad g(r) = a\cdot r$ avec $a = \frac{g_0}{R_N}$

$$\Rightarrow g(r)\cdot r^2 = \frac{g_0}{R_N} r^3 \Rightarrow \frac{d}{dr}(r^2 g(r)) = \frac{3 g_0 r^2}{R_N}$$

$$\Rightarrow \boxed{\rho(r) = \frac{3 g_0 \cancel{r^2}}{4\pi \cancel{r^2} R_N \mathcal{G}}} = \frac{3 g_0}{4\pi \mathcal{G} R_N} = \text{cste} = \rho_0$$

$$= \frac{3\cancel{\mathcal{G}} M_T / R_T^2}{4\pi \cancel{\mathcal{G}} R_N} = \frac{3 M_T}{4\pi R_T^2 R_N}$$

pour $r \in]R_N, R_T[$ $\quad g(r) = g_0$

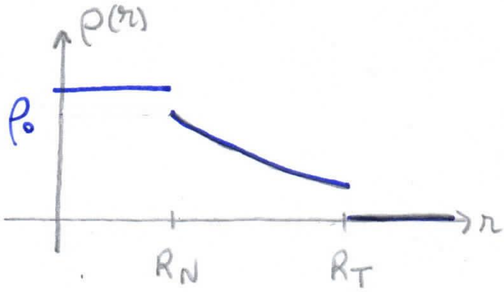
$$\Rightarrow g(r)\cdot r^2 = g_0 \cdot r^2 \Rightarrow \frac{d}{dr}(r^2 g(r)) = 2 g_0 r$$

$$\Rightarrow \boxed{\rho(r) = \frac{\cancel{2} g_0 \cancel{r}}{2\cancel{4}\pi r^{\cancel{2}} \mathcal{G}}} = \frac{g_0}{2\pi \mathcal{G} r}$$

pour $r > R_T$ $\quad g(r) = \frac{b}{r^2}$, b cste

$$\Rightarrow g(r)\cdot r^2 = b \Rightarrow \frac{d}{dr}(r^2 g(r)) = 0$$

$$\Rightarrow \boxed{\rho(r) = 0}$$ ← cohérent : pas de matière pour $r > R_T$!



en $r = R_N$:

$$\begin{cases} \rho(r = R_N^-) = \rho_0 = \frac{3 g_0}{4\pi \mathcal{G} R_N} \\ \rho(r = R_N^+) = \frac{g_0}{2\pi \mathcal{G} R_N} < \rho_0 \end{cases}$$

↳ Terre : planète différenciée :

composit° inhomogène

≠ noyau / manteau + croûte.

(*) Rq : 5/2 → éq° de Maxwell-Gauss : $\text{div}\,\vec{E} = \frac{\rho}{\varepsilon_0}$ → analogue gravitat° : $\text{div}\,\vec{g} = -4\pi\mathcal{G}\rho$

en sphériques : $\text{div}(\vec{g}(M)) = \text{div}(-g(r)\,\vec{u}_r) = \frac{1}{r^2}\cdot\frac{d(r^2(-g(r)))}{dr}$

$\Leftrightarrow +4\pi\mathcal{G}\rho = +\frac{1}{r^2}\cdot\frac{d(r^2 g(r))}{dr}$ → on retrouve l'éq° (*)