Calcul vectoriel

Exercice 1

1) $\vec{x}_0 \cdot \vec{x}_1 = \cos\alpha$ $\quad$ $\vec{x}_1 \cdot \vec{x}_2 = \cos\theta$ $\quad$ $\vec{z}_1 \cdot \vec{x}_2 = \cos\left(\frac{\pi}{2}+\theta\right) = -\sin\theta$

$\vec{x}_1 \cdot \vec{y}_0 = \cos\left(\frac{\pi}{2}-\alpha\right) = \sin\alpha$

2) • $\vec{x}_0 \wedge \vec{y}_1 = \left|\sin\left(\alpha+\frac{\pi}{2}\right)\right| \vec{z}_0 = \cos\alpha\, \vec{z}_0$

le signe se détermine avec la règle du tire-bouchon ou de la main droite.

ou dans $(\vec{x}_0, \vec{y}_0, \vec{z}_0)$: $\begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix} \wedge \begin{vmatrix} -\sin\alpha \\ \cos\alpha \\ 0 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ \cos\alpha \end{vmatrix}$

• $\vec{x}_2 \wedge \vec{z}_1 = -\left|\sin\left(\theta+\frac{\pi}{2}\right)\right| \vec{y}_1 = -\cos\theta\, \vec{y}_1$

ou dans $(\vec{x}_1, \vec{y}_1, \vec{z}_1)$: $\begin{vmatrix} \cos\theta \\ 0 \\ -\sin\theta \end{vmatrix} \wedge \begin{vmatrix} 0 \\ 0 \\ 1 \end{vmatrix} = \begin{vmatrix} 0 \\ -\cos\theta \\ 0 \end{vmatrix}$

• $\vec{y}_0 \wedge \vec{x}_1 = -\left|\sin\left(\frac{\pi}{2}-\alpha\right)\right| \vec{z}_0 = -\cos\alpha\, \vec{z}_0$

ou dans $(\vec{x}_0, \vec{y}_0, \vec{z}_0)$: $\begin{vmatrix} 0 \\ 1 \\ 0 \end{vmatrix} \wedge \begin{vmatrix} \cos\alpha \\ \sin\alpha \\ 0 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ -\cos\alpha \end{vmatrix}$

• $\vec{z}_2 \wedge \vec{x}_1 = \left|\sin\left(\frac{\pi}{2}-\theta\right)\right| \vec{y}_1 = \cos\theta\, \vec{y}_1$

ou dans $(\vec{x}_1, \vec{y}_1, \vec{z}_1)$: $\begin{vmatrix} \sin\theta \\ 0 \\ \cos\theta \end{vmatrix} \wedge \begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix} = \begin{vmatrix} 0 \\ \cos\theta \\ 0 \end{vmatrix}$

Exercice 2

• $\vec{P} \wedge \vec{R}$

Dans $(\vec{e}_x, \vec{e}_y, \vec{e}_z)$: $\begin{vmatrix} -\|\vec{P}\|\sin\theta \\ -\|\vec{P}\|\cos\theta \\ 0 \end{vmatrix} \wedge \begin{vmatrix} \|\vec{R}\|\cos\alpha \\ \|\vec{R}\|\sin\alpha \\ 0 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ \|\vec{P}\|\|\vec{R}\| \underbrace{\left(-\sin\theta\sin\alpha + \cos\alpha\cos\theta\right)} \end{vmatrix}$

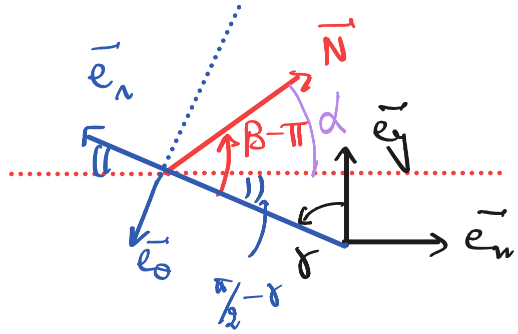
$= \|\vec{P}\|\|\vec{R}\| \cos(\alpha+\theta)$

• $\vec{T} \wedge \vec{e}_x$

Dans $(\vec{e}_x, \vec{e}_y, \vec{e}_z)$: $\begin{vmatrix} -\|\vec{T}\|\cos\gamma \\ \|\vec{T}\|\sin\gamma \\ 0 \end{vmatrix} \wedge \begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ -\|\vec{T}\|\sin\gamma \end{vmatrix}$

• $\vec{e_r} \wedge \vec{N}$:

Dans $(\vec{e_x}, \vec{e_y}, \vec{e_z})$: $\begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix} \wedge \begin{vmatrix} \|\vec{N}\| \cos\alpha \\ \|\vec{N}\| \sin\alpha \\ 0 \end{vmatrix}$



$= \begin{vmatrix} 0 \\ 0 \\ \|\vec{N}\| \sin\alpha \end{vmatrix} = \|\vec{N}\| \cos(\gamma+\beta)$

$\alpha = \beta - \pi - \pi/2 + \gamma = \beta + \gamma - \frac{3\pi}{2}$

Exercice 3

a) $\vec{A} \wedge \vec{B} \begin{vmatrix} 8-15 = -7 \\ 18-4 = 14 \\ 5-12 = -7 \end{vmatrix}$

b) $(\vec{B}+\vec{A}) \wedge \vec{A} = \vec{B} \wedge \vec{A} = -\vec{A} \wedge \vec{B} = \begin{vmatrix} 7 \\ -14 \\ 7 \end{vmatrix}$

c) $\vec{A} \wedge \vec{B}$ selon $\vec{e_x}$ $= -7$

d) $\vec{B} \wedge \vec{e_x} = \begin{vmatrix} 6 \\ 5 \\ 4 \end{vmatrix} \wedge \begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix} = \begin{vmatrix} 0 \\ 4 \\ -5 \end{vmatrix}$

$\vec{A} \cdot (\vec{B} \wedge \vec{e_x}) = 1 \times 0 + 2 \times 4 + 3 \times (-5) = -7$

e) $\vec{B} \wedge \vec{C} = \begin{vmatrix} -9 \\ 6 \\ 6 \end{vmatrix}$

$\vec{A} \wedge (\vec{B} \wedge \vec{C}) = \begin{vmatrix} 1 \\ 2 \\ 3 \end{vmatrix} \wedge \begin{vmatrix} -9 \\ 6 \\ 6 \end{vmatrix} = \begin{vmatrix} -6 \\ -33 \\ 24 \end{vmatrix}$

f) $\vec{A} \cdot \vec{C} = 1 \times 0 + 2 \times 1 + 3 \times (-1) = -1$ $\quad \vec{A} \cdot \vec{B} = 6 + 10 + 12 = 28$

$(\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C} = \begin{vmatrix} -1 \times 6 - 28 \times 0 = -6 \\ -1 \times 5 - 28 \times 1 = -33 \\ -1 \times 4 - 28(-1) = 24 \end{vmatrix}$

Exercice 4

a) $\mathcal{M}_{Oz}(\vec{F}) = -\|\vec{F}\| \times \underbrace{OM\cos\alpha}_{\text{bras de levier}}$ $\quad OM = \ell\sin\alpha \quad$ $\mathcal{M}_{Oz}(\vec{F}) = -\|\vec{F}\|\ell\cos\alpha\sin\alpha$

ou $\mathcal{M}_{Oz}(\vec{F}) = \vec{\mathcal{M}}_O(\vec{F})\cdot\vec{e}_z \quad \vec{\mathcal{M}}_O(\vec{F}) = \begin{vmatrix} 0 \\ OM \\ 0 \end{vmatrix} \wedge \begin{vmatrix} \|\vec{F}\|\cos\alpha \\ -\|\vec{F}\|\sin\alpha \\ 0 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ -\|\vec{F}\| OM\cos\alpha \end{vmatrix}$ ...

b) $\mathcal{M}_{Az}(\vec{F}) = 0$ car la droite d'action de $\vec{F}$ passe par A. (ou $\vec{\mathcal{M}}_A(\vec{F}) = \vec{0}$ car $\vec{F} // \vec{AM}$)

Exercice 5

a) $\vec{\mathcal{M}}_A(\vec{P}) = \vec{AG}\wedge\vec{P} = \begin{vmatrix} -L/2\cos\alpha \\ L/2\sin\alpha \\ 0 \end{vmatrix} \wedge \begin{vmatrix} 0 \\ -mg \\ 0 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ \frac{L}{2}\cos\alpha\, mg \end{vmatrix}$

b) $\vec{\mathcal{M}}_O(\vec{P}) = \vec{OG}\wedge\vec{P} = \begin{vmatrix} \ell - L/2\cos\alpha \\ -R + L/2\sin\alpha \\ 0 \end{vmatrix} \wedge \begin{vmatrix} 0 \\ -mg \\ 0 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ -mg\left(\ell - \frac{L}{2}\cos\alpha\right) \end{vmatrix}$

$\vec{OI} + \vec{IA} + \vec{AG}$

c) $\vec{\mathcal{M}}_I(\vec{P}) = \vec{IG}\wedge\vec{P} = \begin{vmatrix} \ell - L/2\cos\alpha \\ L/2\sin\alpha \\ 0 \end{vmatrix} \wedge \begin{vmatrix} 0 \\ -mg \\ 0 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ -mg\left(\ell - \frac{L}{2}\cos\alpha\right) \end{vmatrix}$

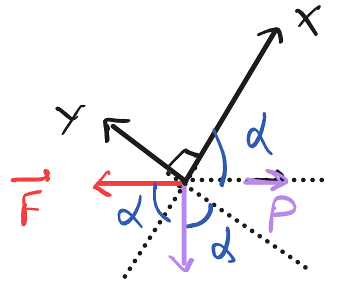
$\vec{IA} + \vec{AG}$

Exercice 6

a) $\vec{OB} = \frac{a}{2}\vec{e}_x + a\vec{e}_y$ $\qquad$ b) $\vec{OG} = \frac{a}{2}\vec{e}_x + \frac{a}{3}\vec{e}_y$

c) $\vec{P} = P\left(-\sin\alpha\,\vec{e}_x - \cos\alpha\,\vec{e}_y\right)$

d) $\vec{F} = F\left(-\cos\alpha\,\vec{e}_x + \sin\alpha\,\vec{e}_y\right)$



e) $\vec{\mathcal{M}}_O(\vec{F}) = \vec{OB}\wedge\vec{F} = \left(F\frac{a}{2}\sin\alpha + Fa\cos\alpha\right)\vec{e}_z$

f) $\vec{\mathcal{M}}_O(m\vec{g}) = \vec{OG}\wedge\vec{P} = \left(-P\frac{a}{2}\cos\alpha + P\frac{a}{3}\sin\alpha\right)\vec{e}_z$

g) équilibre : $\sum\vec{\mathcal{M}}_O = \vec{0}$

$\iff \tan\alpha = \frac{3P - 6F}{3F + 2P}$