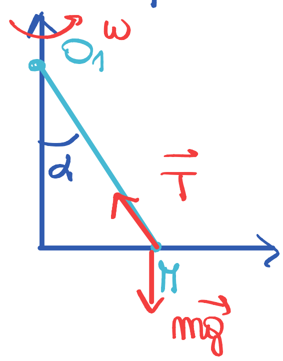
Correction TD16

Exercice 1

Système : {point matériel M} Réf : terrestre supposé galiléen

BdF : $m\vec g$, $\vec T$



1) $\vec L_{O_1} = \overrightarrow{O_1M} \wedge m\vec v$ hyp : $\alpha = C^{te}$

dans $(\vec u_r, \vec u_\theta, \vec u_z)$: $\begin{vmatrix} R \\ 0 \\ -L\cos\alpha \end{vmatrix} \wedge \begin{vmatrix} 0 \\ mR\omega \\ 0 \end{vmatrix} = \begin{vmatrix} mRL\cos\alpha\,\omega \\ 0 \\ mR^2\omega \end{vmatrix}$

2) $\frac{d\vec L_{O_1}}{dt} = \sum \vec M_{O_1} = \vec M_{O_1}(m\vec g)$ $\vec M_{O_1}(\vec T) = \overrightarrow{O_1M} \wedge m\vec g = \vec 0$

$\begin{vmatrix} R \\ 0 \\ -L\cos\alpha \end{vmatrix} \wedge \begin{vmatrix} 0 \\ 0 \\ -mg \end{vmatrix} = \begin{vmatrix} 0 \\ mgR \\ 0 \end{vmatrix}$

$\frac{d\vec L_{O_1}}{dt} = mRL\cos\alpha\,\omega^2\,\vec u_\theta$

$\cancel{m}\cancel{R}L\cos\alpha\,\omega^2 = \cancel{m}g\cancel{R}$

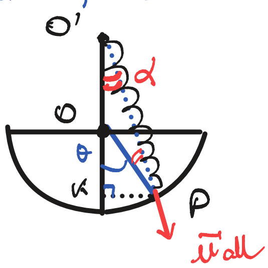
$\boxed{\cos\alpha = \frac{g}{L\omega^2}}$

Exercice 2

Système : {point matériel M} Réf : terrestre supposé galiléen BdF : $m\vec g$, $\vec R_n$, $\vec F_e$

1) $\ell = \left((a+a\cos\theta)^2 + a^2\sin^2\theta\right)^{1/2} = (2a^2+2a^2\cos\theta)^{1/2} = \left(2a^2\underbrace{(1+\cos\theta)}_{2\cos^2\theta/2}\right)^{1/2}$ $\boxed{\ell = 2a\cos\frac{\theta}{2}}$

2) $\vec F_e = -k(\ell-\ell_0)\vec u_{el}$



$\alpha = \frac{\theta}{2}$ (angle au centre = 2 × angle inscrit qui intercepte le même arc)

(autre démo : $\alpha + \frac{\pi}{2} + (\alpha + \frac{\pi}{2} - \theta) = \pi$ dans OKP $\Rightarrow \alpha = \frac{\theta}{2}$)

$\vec M_O(\vec F_e) = \overrightarrow{OP} \wedge -k(\ell-\ell_0)\vec u_{el} = \begin{vmatrix} a\cos\theta \\ a\sin\theta \\ 0 \end{vmatrix} \wedge \begin{vmatrix} -k(\ell-\ell_0)\cos\frac{\theta}{2} \\ -k(\ell-\ell_0)\sin\theta/2 \\ 0 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ -k(\ell-\ell_0)a(-\cos\theta\sin\frac{\theta}{2} + \sin\theta\cos\frac{\theta}{2}) \end{vmatrix}$

Dans $(\vec u_x, \vec u_y, \vec u_z)$

$$\vec{M}_O(\vec{F}_e) = k\left(2a\cos\frac{\theta}{2} - l_0\right)a\sin\frac{\theta}{2}\,\vec{e}_z$$

3) $\dfrac{d\vec{L}_O}{dt} = \vec{M}_O(\vec{F}_e) + \vec{M}_O(m\vec{g}) + \underbrace{\vec{M}_O(\vec{R}_n)}_{\vec{0}\text{ car }\vec{R}_n \perp \vec{OP}}$

$\vec{M}_O(m\vec{g}) = \vec{OP} \wedge m\vec{g} = -mga\sin\theta\,\vec{u}_z$

$\vec{L}_O = \vec{OP} \wedge m\vec{v} = a\vec{e}_r \wedge ma\dot{\theta}\vec{u}_\theta = ma^2\dot{\theta}\,\vec{u}_z$

TMC $\Rightarrow$ $ma^2\ddot{\theta} = -mga\sin\theta + k\left(2a\cos\frac{\theta}{2} - l_0\right)a\sin\frac{\theta}{2}$

$\ddot{\theta} + \frac{g}{a}\sin\theta - \frac{k}{ma}\left(2a\cos\frac{\theta}{2} - l_0\right)\sin\frac{\theta}{2} = 0$

$$\ddot{\theta} + \left(\frac{g}{a} - \frac{k}{m}\right)\sin\theta + \frac{kl_0}{ma}\sin\frac{\theta}{2} = 0$$

4) <u>A l'équilibre $\Sigma\vec{M}_O = \vec{0}$</u>

$k = \dfrac{2mg}{a}$ $\quad l_0 = \dfrac{\sqrt{3}}{2}a$

$$2\underbrace{\left(\frac{g}{a} - \frac{k}{m}\right)}_{-\frac{k}{2m}}\cos\frac{\theta_e}{2}\sin\frac{\theta_e}{2} + \underbrace{\frac{kl_0}{ma}}_{\frac{k\sqrt{3}}{2m}}\sin\frac{\theta_e}{2} = 0$$

$$\frac{k}{m}\sin\frac{\theta_e}{2}\left(\frac{\sqrt{3}}{2} - \cos\frac{\theta_e}{2}\right) = 0$$

$\hookrightarrow$ 3 positions d'équilibre : $\theta_{e_1} = 0$ $\quad \theta_{e_2} = \frac{\pi}{3}$ $\quad \theta_{e_3} = -\frac{\pi}{3}$

5) $\theta = \theta_e + \varepsilon \rightarrow \ddot{\varepsilon} - \dfrac{k}{2m}\sin(\theta_e + \varepsilon) + \dfrac{k\sqrt{3}}{2m}\sin\left(\dfrac{\theta_e + \varepsilon}{2}\right)$

. $\sin(\theta_e + \varepsilon) \simeq \sin\theta_e\cos\varepsilon + \cos\theta_e\sin\varepsilon \simeq \sin\theta_e + \varepsilon\cos\theta_e$

. $\sin\dfrac{\theta_e + \varepsilon}{2} \simeq \sin\dfrac{\theta_e}{2} + \dfrac{\varepsilon}{2}\cos\dfrac{\theta_e}{2}$

$$\ddot{\varepsilon} - \frac{k}{2m}\cos\theta_e\,\varepsilon + \frac{k}{2m}\sqrt{3}\,\frac{\varepsilon}{2}\cos\frac{\theta_e}{2} \underbrace{- \frac{k}{2m}\sin\theta_e + \frac{k\sqrt{3}}{2m}\sin\frac{\theta_e}{2}}_{0\ (\text{équilibre, voir 4})} = 0$$

$$\ddot{\varepsilon} + \frac{k}{2m}\left(\frac{\sqrt{3}}{2}\cos\frac{\theta_e}{2} - \cos\theta_e\right)\varepsilon = 0$$

Si $\frac{\sqrt{3}}{2}\cos\frac{\theta_e}{2} - \cos\theta_e > 0 \rightarrow$ OHNA donc stable sinon instable.

$\theta_{e_1} = 0$ $\quad \frac{\sqrt{3}}{2}\cos\frac{\theta_{e_1}}{2} - \cos\theta_{e_1} = \frac{\sqrt{3}}{2} - 1 < 0$ instable

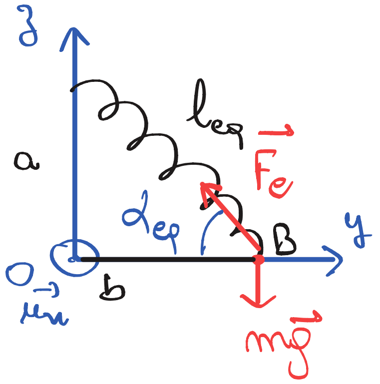
$\theta_{e_2} = \pm\frac{\pi}{3}$ $\quad \frac{\sqrt{3}}{2}\cos\frac{\pi}{6} - \cos\frac{\pi}{3} = \frac{\sqrt{3}}{2}\times\frac{\sqrt{3}}{2} - \frac{1}{2} = 0{,}25 > 0$ stable

Exercice 3

1) Equilibre : $\sum\vec{F} = \vec{0}$ $\quad \sum\vec{M}_O = \vec{0}$

Système : {point B} $\quad$ Réf : terrestre supposé galiléen

BdF : poids $m\vec{g}$, force de rappel élastique $\vec{F_e}$, tension $\vec{T}$ de la tige



• $\vec{M}_O(\vec{T}) = \vec{0}$ $\quad \vec{T}$ inconnue, colinéaire à $\vec{OB}$

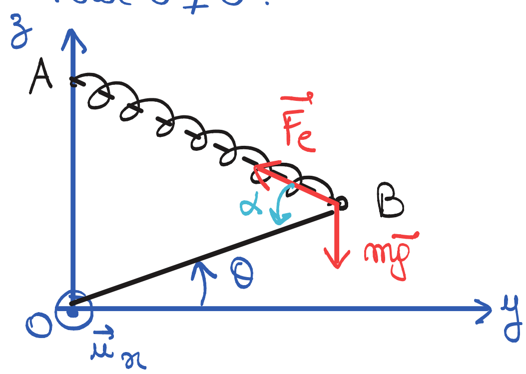
• $\vec{M}_O(m\vec{g}) = b\vec{u}_y \wedge -mg\vec{u}_z = -mgb\,\vec{u}_x$

• $\vec{M}_O(\vec{F_e}) = k(l_{eq} - l_0)\,b\sin\alpha_{eq}\,\vec{u}_x$ $\quad \sin\alpha_{eq} = \frac{a}{l_{eq}}$

$mg\cancel{b} = k\cancel{b}a\left(1 - \frac{l_0}{l_{eq}}\right)$ $\quad l_{eq} = \dfrac{l_0}{1 - mg/ka}$

$l_{eq} > 0 \Rightarrow a > \frac{mg}{k}$

2) Pour $\theta \neq 0$:



$\vec{M}_O(m\vec{g}) = \vec{OB}\wedge m\vec{g} = -mgb\cos\theta\,\vec{u}_x$

$\vec{M}_O(\vec{F_e}) = \vec{OB}\wedge\vec{F_e} = bk(l - l_0)\sin\alpha\,\vec{u}_x$

loi des sinus : $\frac{\sin\alpha}{a} = \frac{\sin(\pi/2 - \theta)}{l}$

$\sin\alpha = a\frac{\cos\theta}{l}$

$\vec{L}_O = \vec{OB}\wedge m\vec{v} = b\vec{e}_r \wedge mb\dot{\theta}\vec{u}_\theta = mb^2\dot{\theta}\,\vec{u}_x$

TMC : $mb^2\ddot{\theta} = -mgb\cos\theta + bk(l - l_0)\frac{a}{l}\cos\theta \simeq -mgb + bka - bkl_0\frac{a}{l}$

$l^2 = a^2 + b^2 - 2ab\sin\theta \simeq l_{eq}^2 - 2ab\theta$ $\quad \cos\theta \simeq 1$ et $\sin\theta \simeq \theta$

$mb^2\ddot{\theta} = -mgb + bka - bkl_0a\left(l_{eq}^2 - 2ab\theta\right)^{-1/2}$ $\quad l_{eq}^2\left(1 - \underbrace{\frac{2ab}{l_{eq}^2}\theta}_{\ll 1}\right)$

$mb^2\ddot{\theta} = -mgb + bka - bkl_0a \times \frac{1}{l_{eq}}\left(1 - 2ab\frac{\theta}{l_{eq}^2}\right)^{-1/2}$

DL à l'ordre 1

$$mb^2\ddot{\theta} \simeq -mgb + bka - bkl_0 a \times \frac{1}{l_{eq}}\left(1 + ab\frac{\theta}{l_{eq}^2}\right) \qquad mg = ka\left(1-\frac{l_0}{l_{eq}}\right)$$

$$m\cancel{b}^2\ddot{\theta} \simeq \underbrace{-\cancel{kab} + \cancel{\frac{kabl_0}{l_{eq}}}} + \cancel{bka} - \cancel{\frac{bkl_0 a}{l_{eq}}} - \cancel{b}kl_0 a^2\frac{\theta}{l_{eq}^3}$$

$$\ddot{\theta} + \frac{k}{m}\frac{l_0 a^2}{l_{eq}^3}\theta = 0 \qquad \text{OHNA} \qquad \omega_0 = \left(\frac{k}{m}\frac{l_0 a^2}{l_{eq}^3}\right)^{1/2}$$ pulsation propre des petites oscillations

$$T_0 = \frac{2\pi}{\omega_0} \Rightarrow T_0 = 2\pi\sqrt{\frac{m l_{eq}^3}{k l_0 a^2}}$$

Exercice 4

1) Système : {e-} Réf : terrestre supposé galiléen BdF : $\vec{F}_{elec}$ (on néglige le poids)

$$\vec{F}_{elec} = \frac{-e^2}{4\pi\varepsilon_0 r^2}\vec{u}_r$$

+e .... e, r, $\vec{u}_r$, $\vec{F}_{elec}$

Sur 1 orbite circulaire $\vec{F}_{elec} \perp \vec{v}$, ne travaille pas donc $\Delta E_c = 0 \rightarrow v = C^{te}$

$$\text{PFD} \rightarrow -m_e\frac{v^2}{r} = -\frac{e^2}{4\pi\varepsilon_0 r^2} \rightarrow v = \left(\frac{e^2}{m_e 4\pi\varepsilon_0 r}\right)^{1/2}$$

2) $E_c = \frac{1}{2}m_e v^2$ $\quad E_p = -\frac{e^2}{4\pi\varepsilon_0 r}$ $\quad E_m = \frac{1}{2}m_e v^2 - \frac{e^2}{4\pi\varepsilon_0 r} = \frac{-e^2}{8\pi\varepsilon_0 r}$

3) $\vec{L}_o = \vec{OM} \wedge m\vec{v} = m r v \vec{u}_z$ $\quad m r v = n\hbar$ $\quad v = \frac{nh}{2\pi r m_e}$

4) $$\frac{e^2}{\cancel{m_e}\cancel{4\pi}\varepsilon_0 \cancel{r}} = \frac{n^2 h^2}{\cancel{4}\pi^2 r^{\cancel{2}} m_e^{\cancel{2}}} \qquad r = n^2\underbrace{\frac{h^2\varepsilon_0}{\pi e^2 m_e}}_{r_0 = 53\text{ pm}}$$

5) $$E_m = \frac{-e^2}{8\pi\varepsilon_0 r} = \frac{-e^4 m_e}{8\varepsilon_0^2 n^2 h^2} \qquad E_0 = \frac{e^4 m_e}{8\varepsilon_0^2 h^2} \qquad E_0 = 13,6\text{ eV}$$

6) $$v_0 = \left(\frac{e^2}{m_e 4\pi\varepsilon_0 r_0}\right)^{1/2} = \left(\frac{e^2}{4\cancel{\pi}\varepsilon_0 \cancel{m_e}} \times \frac{\cancel{\pi} e^2 \cancel{m_e}}{h^2\varepsilon_0}\right)^{1/2} = \left(\frac{e^4}{4\varepsilon_0^2 h^2}\right)^{1/2}$$

(+ rapide en remarquant $E_c = -E_m$)

$\frac{1}{2}mv^2$ $\quad \frac{E_0}{n^2}$

$v_0 = 2,2 \times 10^6\text{ m.s}^{-1} < 0,1c$ (non relativiste)

Energie d'ionisation : $\Delta E = E_\infty - E_1 = -E_1 = E_0 = 13,6\text{ eV}$

E_∞ : 0

7) $$h\frac{c}{\lambda_{n\to m}} = E_n - E_m \quad (\text{car } E_n > E_m) \qquad \frac{1}{\lambda_{n\to m}} = \underbrace{\frac{E_0}{hc}}_{R_H}\left(\frac{1}{m^2} - \frac{1}{n^2}\right)$$

8) $E_n = \frac{-E_0}{n^2} = E_1 + \Delta E \rightarrow n = \left(\frac{-E_0}{-E_0 + \Delta E}\right)^{1/2} = 2$ n entier : l'e⁻ passe sur l'orbite $n = 2$

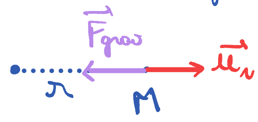
9) le photon émis a 1 énergie de 10,2 eV

$\nu = \frac{E_\gamma}{h}$ $\lambda = \frac{c}{\nu} = 121 \text{ nm} \rightarrow$ UV

Exercice 5

1) Système : {Satellite} Réf : géocentrique supposé galiléen BdF : $\vec{F}_{grav} = -\frac{M_T m}{r^2}\vec{u}_r$

TMC : $\frac{d\vec{L}_0}{dt} = \vec{OM} \wedge \vec{F}_{grav} = \vec{0}$
$\vec{L} \,/\!/\, \vec{OM}$ (force centrale)



$\vec{L_0} = \vec{Cte}$ or $\vec{L_0} = \vec{OM} \wedge m\vec{v}$ $\vec{OM}$ et $\vec{v}$ ⊥ à $\vec{L_0}$ par propriété du produit vectoriel

$\vec{OM}$ et $\vec{v}$ ∈ à 1 plan ⊥ à $\vec{L_0} = \vec{Cte}$ → le mouvement est plan, le plan de la trajectoire contient le point O

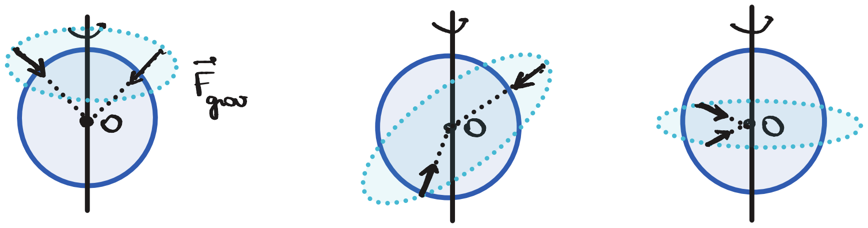
2) Orbite circulaire telle que $r = R + z$ PFD ⇒ $m\frac{v^2}{r} = G\frac{M_T m}{r^2} \rightarrow v = \left(\frac{GM_T}{R+z}\right)^{1/2} = 7{,}6 \text{ km.s}^{-1}$

3) $E_m = \frac{1}{2}mv^2 - G\frac{mM_T}{r} = -\frac{GmM_T}{2r}$

4) $T = \frac{2\pi r}{v} = 2\pi r\left(\frac{r}{GM_T}\right)^{1/2}$ $T^2 = \frac{4\pi^2}{GM_T}r^3$ 3° loi de Kepler

5) Satellite géostationnaire : fixe pour 1 observateur terrestre
trajectoire circulaire donc uniforme car $\vec{F}_{grav} \perp \vec{v} \Rightarrow \Delta E_c = 0$

D'après 1), le plan de la trajectoire ⊥ $\vec{L_0}$ et contient O (centre de la Terre)



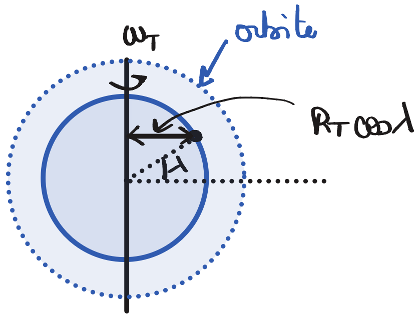
le plan de la trajectoire ne contient pas O

pas géostationnaire

T = période sidérale de la Terre = 86 164 s $T_{sid}^2 = \frac{4\pi^2}{GM_T}(R_T + z)^3 \rightarrow z \simeq 35820 \text{ km}$

6) $\Delta E_m = E_m(\text{orbite}) - E_m(\text{sol}) = -\frac{GmM_T}{r} - \left(\frac{1}{2}m v_{sol}^2 - \frac{GmM_T}{R_T}\right)$

$v_{sol} = \frac{1}{2} m \, R_T \cos\lambda \, \omega_T^2$



$\omega_T = \frac{2\pi}{T_{sid}}$

$$E_m = -G m M_T \left(\frac{1}{\underset{\uparrow R_T + z}{r}} - \frac{1}{R_T} \right) - \frac{1}{2} m R_T \cos\lambda \, \omega_T^2$$

+ faible aux basses latitudes

7) "libération" → cas limite $E_m = 0$: satellite à l'∞ donc $E_p = 0$) et $v_\infty = 0$ (cas limite)

↑ pour échapper à l'attraction terrestre

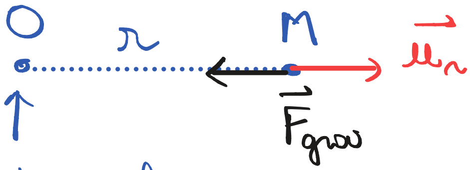
$$\underbrace{\frac{1}{2} m v_{lib}^2 - \frac{G m M_T}{R_T}}_{E_m \text{ (sol)}} = 0$$

$$v_{lib} = \left(\frac{2 G M_T}{R_T} \right)^{1/2}$$

AN : $11{,}2 \text{ km.s}^{-1}$

Exercice 6

1) $\vec{F}_{grav} = -G \frac{m M_T}{r^2} \vec{u}_r$



centre de la Terre

$$\vec{F}_{grav} = -\overrightarrow{grad}\, E_p = \begin{vmatrix} -\frac{\partial E_p}{\partial r} \\ -\frac{1}{r}\frac{\partial E_p}{\partial \theta} \\ -\frac{1}{r\sin\theta}\frac{\partial E_p}{\partial \varphi} \end{vmatrix} = \begin{vmatrix} -G\frac{m M_T}{r^2} \\ 0 \\ 0 \end{vmatrix}$$

→ $E_p = -\frac{G m M_T}{r} + Cte$

⟹ E_p ne dépend que de r

$E_p(\infty) = 0$ → $Cte = 0$

$$E_p = -\frac{G m M_T}{r}$$

2) PFD selon $\vec{u}_r$: $-m \frac{v^2}{r} = -G \frac{m M_T}{r^2}$ → $v = \left(\frac{G M_T}{r} \right)^{1/2}$ $T = \frac{2\pi r}{v}$

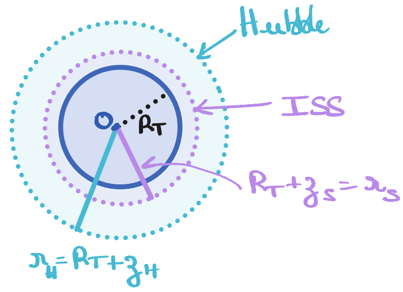
$$T^2 = \frac{4\pi^2}{G M_T} r^3$$

3° loi de Kepler

3) $E_m = \frac{1}{2} m v^2 + E_p = \frac{1}{2} \frac{G m M_T}{r} - \frac{G m M_T}{r}$ ⟹ $E_m = -\frac{G m M_T}{2 r}$

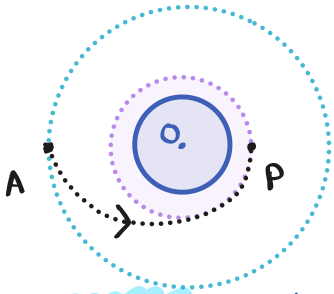
4) $\dfrac{4\pi^2}{GM_T} = \dfrac{T_S^2}{r_S^3} = \dfrac{T_H^2}{r_H^3}$ $\qquad T_S = T_H\left(\dfrac{R_T+z_S}{R_T+z_H}\right)^{3/2}$

AN : $T_S = 93$ min



$v_H = \dfrac{2\pi r_H}{T_H} = 7{,}6.10^3\ m.s^{-1}$ $\qquad v_S = \dfrac{2\pi r_S}{T_S} = 7{,}7.10^3\ m.s^{-1}$

5)



.... orbite de transfert

6) $E_m = -\dfrac{GmM_T}{r_S+r_H}$ (résultat admis pour 1 orbite elliptique) cf cours pour la démo.

7) $E_m(A) = \dfrac12 \cancel{m} v_A^2 - \dfrac{G\cancel{m}M_T}{r_H} = -G\dfrac{\cancel{m}M_T}{r_S+r_H} \rightarrow v_A = \sqrt{2GM_T}\left(\dfrac{1}{r_H} - \dfrac{1}{r_S+r_H}\right)^{1/2}$

or $T^2 = \dfrac{4\pi^2}{GM_T} r^3 \rightarrow$ $\qquad GM_T = \dfrac{4\pi^2}{T_H^2} r_H^3$ $\qquad v_A = \left(\dfrac{8\pi^2}{T_H^2} r_H^3 \dfrac{r_S}{r_H(r_S+r_H)}\right)^{1/2}$

$v_A = 2\sqrt{2}\pi r_H \dfrac{1}{T_H}\left(\dfrac{r_S}{(r_S+r_H)}\right)^{1/2}$ $\qquad v_A = 7{,}5.10^3\ m.s^{-1}$

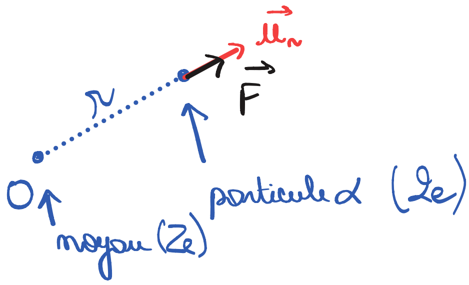
De même : $v_P = 2\sqrt{2}\pi r_S \dfrac{1}{T_S}\left(\dfrac{r_H}{(r_S+r_H)}\right)^{1/2}$ $\qquad v_P = 7{,}7.10^3\ m.s^{-1}$

8) Durée du voyage $\Delta t = \dfrac{T}{2}$, orbite elliptique $T^2 = \dfrac{4\pi^2}{GM_T}\left(\dfrac{r_H+r_S}{2}\right)^3$

$\Delta t = \dfrac12\left(\dfrac{T_H^2}{r_H^3}\left(\dfrac{r_H+r_S}{2}\right)^3\right)^{1/2}$ $\qquad \Delta t = \dfrac{T_H}{4\sqrt{2}}\left(1+\left(\dfrac{r_S}{r_H}\right)\right)^{3/2} = 47$ min

Exercice 7

1) $\vec{F} = \dfrac{2Ze^2}{4\pi\varepsilon r^2}\vec{u}_r$



$$\vec{F} = -\overrightarrow{grad}\,E_p = \boxed{\begin{vmatrix} -\partial E_p/\partial r \\ -1/r\ \partial E_p/\partial\theta \\ -1/r\sin\theta\ \partial E_p/\partial\varphi \end{vmatrix} = \begin{vmatrix} 2Ze^2/4\pi\varepsilon_0 r^2 \\ 0 \\ 0 \end{vmatrix}} \rightarrow E_p = \frac{Ze^2}{2\pi\varepsilon_0 r} + Cte$$

$E_p(r)$

$E_p(\infty) = 0$ $\quad Cte = 0$ $\qquad E_p(r) = \dfrac{Ze^2}{2\pi\varepsilon_0 r}$

2) $\vec F$ seule force appliquée, conservative donc $E_m = C^{te}$

$$E_m = E_m(0) = \frac{1}{2} m v_0^2$$

3) $\vec F$ force centrale $\vec M_O(\vec F) = \vec 0$ TMC $\Rightarrow \vec L_O = \vec{C^{te}}$

$$\vec L_O = \vec L_O(0) = \vec{OM_0} \wedge m\vec v_0 = \begin{vmatrix} OH \\ b \\ 0 \end{vmatrix} \wedge \begin{vmatrix} -mv_0 \\ 0 \\ 0 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ mbv_0 \end{vmatrix} \rightarrow \vec L_O = mbv_0 \vec u_z$$

(cartésien)

4) $$\vec L_O = \begin{vmatrix} r \\ 0 \\ 0 \end{vmatrix} \wedge \begin{vmatrix} m\dot r \\ mr\dot\theta \\ 0 \end{vmatrix} = m r^2 \dot\theta \vec u_z$$

(cylindrique)

5) $$E_m = \frac{1}{2} m \left(\dot r^2 + (r\dot\theta)^2\right) + \frac{Ze}{2\pi\varepsilon r}$$ or $m r^2 \dot\theta = mbv_0 \rightarrow \dot\theta = \frac{bv_0}{r^2}$

$$E_m = \frac{1}{2} m \dot r^2 + \underbrace{\frac{1}{2} m \frac{b^2 v_0^2}{r^2} + \frac{Ze^2}{2\pi\varepsilon r}}_{}$$

énergie potentielle effective $E_{p,eff}(r)$

6) $\dot r_{min} = 0$ $\quad E_m(r_{min}) = E_{p,eff}(r_{min}) = \frac{1}{2} m v_0^2$

$$\frac{1}{2} m \frac{b^2 v_0^2}{r_{min}^2} + \frac{Ze^2}{2\pi\varepsilon r_{min}} = \frac{1}{2} m v_0^2 \qquad \times r_{min}^2$$

$$-\frac{mb^2v_0^2}{2} - \frac{Ze^2}{2\pi\varepsilon} r_{min} + \frac{mv_0^2}{2} r_{min}^2 = 0$$

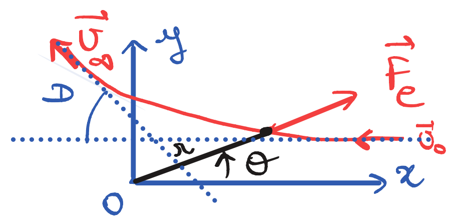
$$\Delta = \left(\frac{Ze^2}{2\pi\varepsilon}\right)^2 + \cancel{4} \frac{m^2 b^2 v_0^4}{\cancel{4}} > 0 \rightarrow \text{2 solutions} \in \mathbb{R}.$$

$r_{min} > 0$ $$r_{min} = \frac{\overbrace{\frac{Ze^2}{2\pi\varepsilon}}^{K} + \sqrt{\Delta}}{mv_0^2} = \frac{K}{mv_0^2} + \frac{1}{mv_0^2}\left(\sqrt{K^2 + m^2 b^2 v_0^4}\right)$$

$$r_{min} = \frac{K}{mv_0^2}\left(1 + \sqrt{1 + \left(\frac{mbv_0^2}{K}\right)^2}\right)$$

7) $$m \frac{dv_x}{dt} = \frac{Ze^2}{2\pi\varepsilon r^2} \cos\theta$$ TMC $\Rightarrow m r^2 \dot\theta = mbv_0$

$$m \frac{dv_x}{dt} = \frac{K}{bv_0} \cos\theta \frac{d\theta}{dt} \rightarrow \int_{-v_0}^{v_{\infty x}} dv_x = \frac{K}{mbv_0} \int_{\theta(0)}^{\theta_\infty} \cos\theta \, d\theta$$



- à $t=0$: particule à l'∞ $\quad \theta = 0$
- à $t \to \infty$: " " $\quad \theta \to \pi - D$

Conservat° de $E_m \Rightarrow v_\infty = v_0$, selon Ox $v_{\infty x} = -v_0 \cos D$

$$\underbrace{-v_0\cos D + v_0}_{v_0(1-\cos D)} = \frac{K}{m b v_0}\,\underbrace{\sin D}_{2\cos\frac{D}{2}\sin\frac{D}{2}} \qquad 1-\cos D = 2\sin^2\frac{D}{2} \Rightarrow \qquad \tan\frac{D}{2} = \frac{K}{m b v_0^2}$$

8) $$b = \frac{Ze^2}{2\pi\varepsilon_0 m v_0^2 \tan D/2}$$

$D_1 = 60° \to b_1 = 1{,}06 \times 10^{-14}$ m $\to r_{min,1} = 1{,}62.10^{-14}$ m

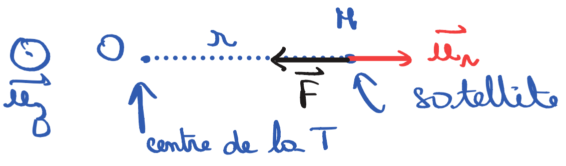
$D_2 = 180° \to b_2 = 0$ m $\to r_{min,2} = 1{,}23.10^{-14}$ m

rayon de l'atome d'or $< r_{min} \to$ ordre de grandeur : 10^{-14} m

Exercice 8

1) Système : {satellite} Réf : terrestre supposé galiléen BdF : force de gravitation

$\vec{F} = -G\frac{mM_T}{r^2}\vec{u_r}$ force centrale $\to$ TMC $\Rightarrow \vec{L_O} = \vec{Cte}$



$$\vec{L_O} = \vec{OM} \wedge m\vec{v} = \begin{vmatrix} r \\ 0 \\ 0 \end{vmatrix} \wedge \begin{vmatrix} 0 \\ r\omega \\ 0 \end{vmatrix}$$ orbite circulaire ($r = $ Cte)

$$\vec{L_O} = m r^2 \omega \vec{u_z} = \vec{Cte}$$

Donc $C = r^2\omega = $ Cte   Mouvement circulaire : $\vec{v} = R\omega\vec{u_\theta}$ $\|\vec{v}\| = \frac{C}{R} = $ Cte

2) PFD selon $\vec{u_r} \Rightarrow -m\frac{v^2}{r} = -G\frac{mM_T}{r^2}$ $v = \left(\frac{GM_T}{r}\right)^{1/2}$

AN : v (1000 km) $= 7{,}4 \times 10^3$ km.s^{-1}

3) $E_c = \frac{1}{2}mv^2 = \frac{mGM_T}{2r}$ $E_p = -G\frac{mM_T}{r}$ $E_m = -\frac{GmM_T}{2r}$

4) TPM : $\frac{dE_m}{dt} = \mathcal{P}(\vec{f})$

$$GmM_T\frac{\dot r}{2r^2} = -\lambda m v^3$$

$\vec{v} = \dot r\vec{u_r} + r\dot\theta\vec{u_\theta} \simeq \sqrt{\frac{GM_T}{r}}\,\vec{u_\theta}$ (descente lente $\dot r\vec{u_r}$ négligée)

$$GM_T\frac{\dot r}{2r^2} = -\lambda\left(\frac{GM_T}{r}\right)^{3/2}$$

$$\dot r = -2\lambda\sqrt{GM_T}\sqrt{r}$$

5) $\dot r < 0$ donc le satellite descend mais si $r \searrow$ $E_c \nearrow$, la vitesse augmente !

(ou $E_m \searrow$ car $\mathcal{P}(\vec{f}) < 0$, $E_m = -\frac{GmM_T}{2r}$ si $E_m \searrow$: $r \searrow$)

6) $E_c = \frac{GmM_T}{2r(t)} \nearrow$ $E_p = -\frac{GmM_T}{r(t)} \searrow$

Quand r diminue, la force de gravitation travaille : son travail est positif (E_p décroit) $\Delta E_c = \underbrace{-\Delta E_p}_{>0} + \underbrace{W(\vec{f})}_{<0}$ si $|\Delta E_p| > |W(\vec{f})|$ alors $\Delta E_c > 0$ et $v \nearrow$.

Au total c'est E_m qui $\searrow$.

7) $\frac{\dot{r}}{2\sqrt{r}} = -\lambda\sqrt{M_T G}$ on intègre entre $r(0)$ et $r(t)$: $\sqrt{r(t)} - \sqrt{r(0)} = -\lambda\sqrt{GM_T}\, t$

$$\sqrt{r(t)} = \sqrt{r(0)} - \lambda\sqrt{GM_T}\, t$$

étude sur 1 jour : $r(0) = 7400$ km $r(1j) = (7400 - 2\times10^{-3})$ km

$$\lambda = \frac{\sqrt{r(0)} - \sqrt{r(1j)}}{\sqrt{GM_T}\,\underbrace{\Delta t}_{24\times3600}} = 2\times10^{-16}\ m^{-1}$$

Exercice 9

1) Système : {satellite} Réf : géocentrique supposé galiléen BdF : force de gravitation

$\vec{F} = -G\frac{mM_T}{r^2}\vec{u}_r$ force centrale conservative : $E_m = C^{te}$

TMC $\Rightarrow \vec{L_O} = \vec{C^{te}}$

$$\frac{1}{2}mv_0^2 = \frac{1}{2}m\,v(r_{min})^2 - \frac{GmM_T}{r_{min}} \quad (1)$$

$\vec{L_O} = \underbrace{\vec{OM}\wedge m\vec{v}}_{mr^2\dot{\theta}\vec{u_z}} = \underbrace{\begin{vmatrix} b \\ \infty \\ 0 \end{vmatrix} \wedge \begin{vmatrix} 0 \\ mv_0 \\ 0 \end{vmatrix}}_{\text{à } t=0 \text{ (ds } (\vec{u_x}, \vec{u_y}, \vec{u_z}))} = mv_0 b\vec{u_z}$ $r^2\dot{\theta} = v_0 b = \frac{L_O}{m}$ (2)

$v^2 = \dot{r}^2 + (r\dot{\theta})^2$ $\dot{r} = 0$ en r_{min} (point P)

(1) $\rightarrow$ $\frac{1}{2}\cancel{m}v_0^2 = \frac{1}{2}\cancel{m}\frac{v_0^2 b^2}{r_{min}^2} - \frac{G\cancel{m}M_T}{r_{min}}$

$$v_0^2 = \frac{\frac{b^2 v_0^2}{r_{min}^2}}{2} - 2\frac{GM_T}{r_{min}} \qquad \times r_{min}^2 \rightarrow v_0^2 r_{min}^2 + 2GM_T r_{min} - b^2 v_0^2 = 0$$

$$\Delta = 4(GM_T) + 4b^2 v_0^4 > 0 \qquad r_{min} > 0 \qquad r_{min} = -\frac{GM_T}{v_0^2} + \frac{1}{2v_0^2}\sqrt{\Delta}$$

$$r_{min} = -\frac{GM_T}{v_0^2} + \sqrt{b^2 + \frac{(GM_T)^2}{v_0^4}}$$

2) $r_{min} > R_T$ $\qquad b_{min} = \left(-\frac{(GM_T)^2}{v_0^4} + \left(R_T + \frac{GM_T}{v_0^2}\right)^2\right)^{1/2} = \left(R_T^2 + 2R_T\frac{GM_T}{v_0^2}\right)^{1/2}$

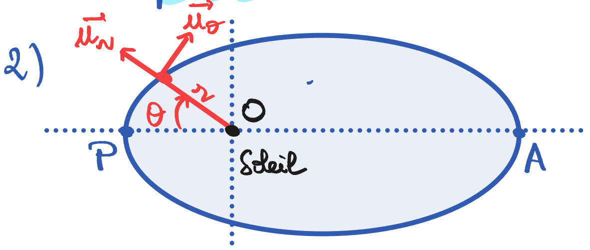
Exercice 10

1) La comète n'est soumise qu'à la force de gravitation, force centrale.

$\vec{M}_0 = \vec{OM} \wedge \vec{F} = \vec{0}$ $\qquad$ TMC : $\frac{d\vec{L_0}}{dt} = \vec{0}$ $\qquad \vec{L_0} = \vec{OM} \wedge m\vec{v} = \vec{Cte}$.

Propriété du produit vectoriel : $\vec{OM}$ et $\vec{v} \perp \vec{L_0}$ donc $\vec{OM}$ et $\vec{v}$ $\in$ à tout instant t au plan contenant le centre de force O et $\perp$ à $\vec{L_0}$.

2)



$OA = r_A = \frac{p}{1-e}$ $\qquad OP = r_p = \frac{p}{1+e}$

$p = r_A(1-e)$ ou $p = r_p(1+e)$ $\qquad p = 1,16$ ua

3) 3° loi de Kepler : $\frac{T^2}{a^3} = \frac{4\pi^2}{GM_S}$ $\qquad a = \frac{r_p + r_A}{2}$ $\rightarrow M_S = 2,0 \times 10^{30}$ kg

4) $E_m = \frac{1}{2}mv^2 - \frac{GM_S m}{r} = -\frac{GM_S m}{2a} \rightarrow v = \left(GM_S\left(\frac{2}{r} - \frac{1}{a}\right)\right)$

en r_p : $v_p = 54,5$ km.s^{-1} $\qquad$ idem en r_A : $v_A = 911$ m.s^{-1}

ou $\frac{dA}{dt} = \frac{1}{2}C$ $\qquad$ C constante des aires $\qquad C = \frac{L_0}{m} = r^2\dot{\theta}$ (constante car $\vec{L_0} = \vec{Cte}$)

vitesse aréolaire $\qquad \frac{dA}{dt} = \frac{S}{T}$ $\qquad$ or en r_A et r_p $\dot{r} = 0$ donc $v_A = r_A\dot{\theta}_A$ et $v_p = r_p\dot{\theta}_p$

$C = r_A r_A \dot\theta_A = r_p r_p \dot\theta_p \Rightarrow r_A v_A = r_p v_p = C$

$\frac{S}{T} = \frac{1}{2} r_A v_A \rightarrow v_A = \frac{2S}{T r_A} = \dots \qquad v_p = \frac{r_A v_A}{r_p} = \dots$

4) a) θ_0 et $-\theta_0$ correspondent à $r = r_T = 1$ ua.

$r_T = \frac{p}{1 + e\cos\theta_0} \qquad \boxed{\theta_0 = \arccos\left(\frac{p - r_T}{e r_T}\right)} \qquad$ AN : $\boxed{\theta_0 \simeq 80{,}5°}$

b) $C = r^2\dot\theta = \frac{L_0}{m} = Cte \qquad \vec{L_0} = \vec{OA} \wedge m\vec{v_A}$ (ou $\vec{OP} \wedge m\vec{v_p}$)

$= \begin{vmatrix} r_A \\ 0 \\ 0 \end{vmatrix} \wedge \begin{vmatrix} 0 \\ m v_A \\ 0 \end{vmatrix} = \underbrace{m r_A v_A}_{L_0} \vec{u_z}$

$C = r_A v_A$ (ou $r_p v_p$) et $p = r_A(1-e)$

$\frac{r_A^2 (1-e)^2}{(1+e\cos\theta)^2} \frac{d\theta}{dt} = r_A v_A$

$$\boxed{\Delta t = \frac{r_A(1-e)^2}{v_A} \int_{-\theta_0}^{\theta_0} \frac{1}{(1+e\cos\theta)^2} d\theta}$$

AN : $e \simeq 1 \Rightarrow \boxed{\Delta t = 77 j}$

Exercice 10

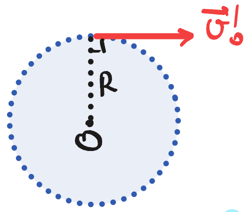
1) Système : {satellite} Réf : géocentrique supposé galiléen

BdF : force de gravitation

Orbite circulaire : PFD $\Rightarrow -m\frac{v_0^2}{R} = -G\frac{m M_T}{R^2} \rightarrow \boxed{v_0 = \sqrt{\frac{G M_T}{R}}}$

$E_m = E_c + E_p = \frac{1}{2} m v_0^2 - G\frac{m M_T}{R} = \boxed{-\frac{G m M_T}{2R}}$

3° loi de Kepler : $\boxed{T_0^2 = \frac{4\pi^2}{G M_T} R^3}$

2) 

O : centre la Terre

$E_m = Cte$ car système conservatif

$$E_m = \frac{1}{2} m v_0'^2 - G\frac{m M_T}{R} \quad \text{or} \quad \frac{GM_T}{R} = v_0^2$$

$$E_m = m\left(\frac{v_0'^2}{2} - v_0^2\right)$$

3) • $E_m < 0$: elliptique $\frac{v_0'}{v_0} < \sqrt{2}$

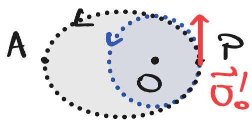
• $E_m = 0$: parabolique $v_0' = \sqrt{2}\, v_0$

• $E_m > 0$: hyperbolique $\frac{v_0'}{v_0} > \sqrt{2}$

4) Erreur faible : $\frac{v_0'}{v_0} < \sqrt{2} \to$ ellipse.

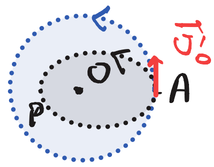
$\vec{v}_0'$ toujours orthoradial donc $\dot{r} = 0 \to r = R$ apogée ou périgée.

• $\Delta v > 0$: $r = R$ périgée $r_{min} = R$ $r_{max} = 2a - R$



$v_0' > v_0$

• $\Delta v < 0$: $r = R$ apogée $r_{max} = R$ $r_{min} = 2a - R$



$v_0' < v_0$

• $E_m = \cancel{m}\left(\frac{v_0'^2}{2} - v_0^2\right) = -\frac{GM_T \cancel{m}}{2a} \to a = \frac{-GM_T}{2v_0^2\left(\frac{v_0'^2}{2v_0^2} - 1\right)}$ $GM_T = v_0^2 R$

$$a = \frac{R}{2 - \left(\frac{v_0'}{v_0}\right)^2}$$

$v_0' = v_0 + \Delta v$ et $\Delta v \ll v_0$ $\left(\frac{v_0'}{v_0}\right)^2 = \left(1 + \frac{\Delta v}{v_0}\right)^2$

$\simeq 1 + 2\frac{\Delta v}{v_0}$ (DL)

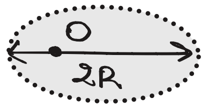
$$a \simeq \frac{R}{1 - 2\frac{\Delta v}{v_0}} \simeq R\left(1 + 2\frac{\Delta v}{v_0}\right) \quad \text{(DL)}$$

• $T^2 = \frac{4\pi^2}{GM_T} a^3 = T_0^2 \times \frac{1}{\left(2 - \left(\frac{v_0'}{v_0}\right)^2\right)^3}$

$$T \simeq T_0\left(1 + 3\frac{\Delta v}{v_0}\right) \quad \text{(DL)}$$

5) Cette fois-ci, comme $\|\vec{v}_0\| = \|\vec{v}_0'\|$ E_m est inchangée $E_m = -\frac{GmM_T}{2R} < 0 \to$ ellipse

Pour 1 ellipse : $E_m = -\frac{GmM_T}{2a} \Rightarrow a = R \to r_{min} + r_{max} = 2R$



3° loi de Képler $\Rightarrow T = T_0$

Pour déterminer r_{min} et r_{max} on peut se servir de $\vec{L}_0 = \vec{Cte}$.

$$\vec{L}_0 = \vec{OM} \wedge m\vec{v}_0' = \begin{vmatrix} R \\ 0 \\ 0 \end{vmatrix} \wedge \begin{vmatrix} m v_0' \sin\alpha \\ m v_0' \cos\alpha \\ 0 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ mRv_0'\cos\alpha \end{vmatrix}$$

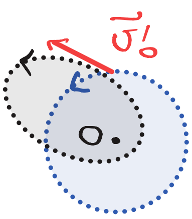
$$L = m r^2 \dot\theta = mRv_0'\cos\alpha \qquad E_m = -\frac{GmM_T}{2R} = \frac{1}{2}m(\underbrace{\dot r^2}_{0 \text{ en A et P}} + (r\dot\theta)^2) - G\frac{mM_T}{r}$$

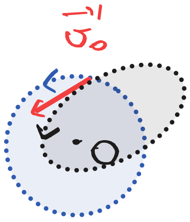
$$-\frac{GmM_T}{2R} = \frac{L_0^2}{2mr^2} - G\frac{mM_T}{r} \quad \text{en } \underbrace{A}_{r_{max}} \text{ et } \underbrace{P}_{r_{min}}$$

$$-G\frac{\cancel{m}M_T}{2R} = \frac{m^{\cancel{2}}R^2v_0^2\cos^2\alpha}{2\cancel{m}r^2} - G\frac{\cancel{m}M_T}{r} \to G\frac{M_T}{2R}r^2 - GM_T r + \frac{R^2v_0^2\cos^2\alpha}{2} = 0$$

$$\Delta = (GM_T)^2 - \cancel{4}\frac{\cancel{GM_T}}{\cancel{2}\cancel{R}} \times \frac{R^{\cancel{2}}\cancel{v_0^2}\cos^2\alpha}{\cancel{2}} = (GM_T)^2(1-\cos^2\alpha) = (GM_T)^2\sin^2\alpha$$

$$r = \frac{GM_T \pm \sqrt{\Delta}}{GM_T}R = R(1 \pm |\sin\alpha|)$$





$r_{max} = R(1 + |\sin\alpha|)$ $\quad r_{min} = R(1 - |\sin\alpha|)$

Exercice 11

1) $\vec{v} = \dot r\,\vec{u}_r + r\dot\theta\,\vec{u}_\theta + \dot z\,\vec{u}_z \qquad \dot z = \frac{2}{a}r\dot r$

$$\vec{L}_0 = \begin{vmatrix} r \\ 0 \\ \frac{r^2}{a} \end{vmatrix} \wedge \begin{vmatrix} m\dot r \\ mr\dot\theta \\ \frac{2m}{a}r\dot r \end{vmatrix} = \begin{vmatrix} -2mr^3\dot r/a^2 \\ -mr^2\dot r/a \\ mr^2\dot\theta \end{vmatrix}$$

2) <u>Système</u> : {bille} <u>Réf</u> : terrestre supposé galiléen

<u>BdF</u> : • poids : $m\vec{g}$

• $\vec{R}_m = R_{m_r}\vec{u}_r + R_{m_z}\vec{u}_z$

$$\vec{M}_O = \vec{OM}\wedge(m\vec{g}+\vec{R}_m) = \begin{vmatrix} r\\ 0\\ z\end{vmatrix} \wedge \begin{vmatrix} R_{m_r}\\ 0\\ R_{m_z}-mg\end{vmatrix} = \begin{vmatrix} 0\\ R_m z - r(R_{m_z}-mg)\\ 0\end{vmatrix}$$

TMC : $\frac{dL_{oz}}{dt} = 0 \rightarrow \boxed{L_{oz} = Cte}$ $\boxed{m r^2\dot{\theta} = Cte}$

3) $E_c = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{r}^2 + (r\dot{\theta})^2 + \dot{z}^2) = \boxed{\frac{1}{2}m\left(\dot{r}^2 + \frac{L_{oz}^2}{mr^2} + \left(\frac{2r\dot{r}}{a}\right)^2\right)}$

4) $m\vec{g}$ est une force conservative $E_p = mgz + \underset{0}{\underset{\sim}{Cte}}$ $\boxed{E_p = mg\frac{r^2}{a}}$

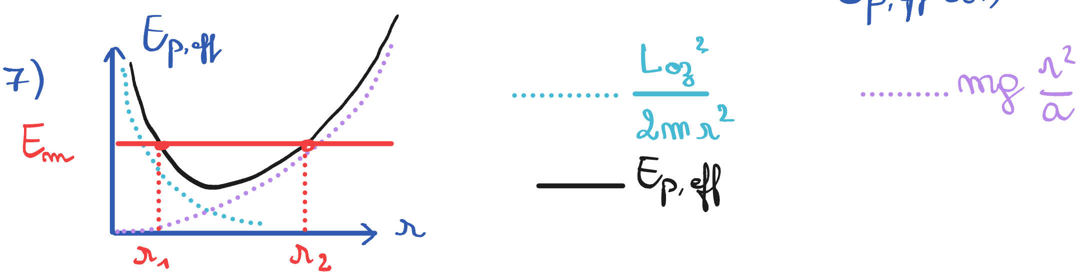
car $E_p(0) = 0$

5) TEM : $\Delta E_m = W^{nc} = 0$ $\boxed{E_m = Cte}$

6) $E_m = Cte \iff \frac{1}{2}m\dot{r}^2 + \frac{L_{oz}^2}{2mr^2} + \frac{1}{2}m\dot{r}^2\left(\frac{4r^2}{a^2}\right) + mg\frac{r^2}{a} = Cte$

$$\boxed{\underbrace{\frac{1}{2}m\dot{r}^2\left(1+4\frac{r^2}{a^2}\right)}_{G(r)} + \underbrace{\frac{L_{oz}^2}{2mr^2} + mg\frac{r^2}{a}}_{E_{p,eff}(r)} = Cte}$$

7)



8) Domaines accessibles à la trajectoire : $E_m \geq E_{p,eff}$

La particule évolue entre 2 disques de rayon r_1 et r_2 tels que

$E_{p,eff}(r_1) = E_{p,eff}(r_2) = E_m \leftarrow$ constante (CI)