Merci à Gabrielle.

TD 15 - Exercice n°3 :

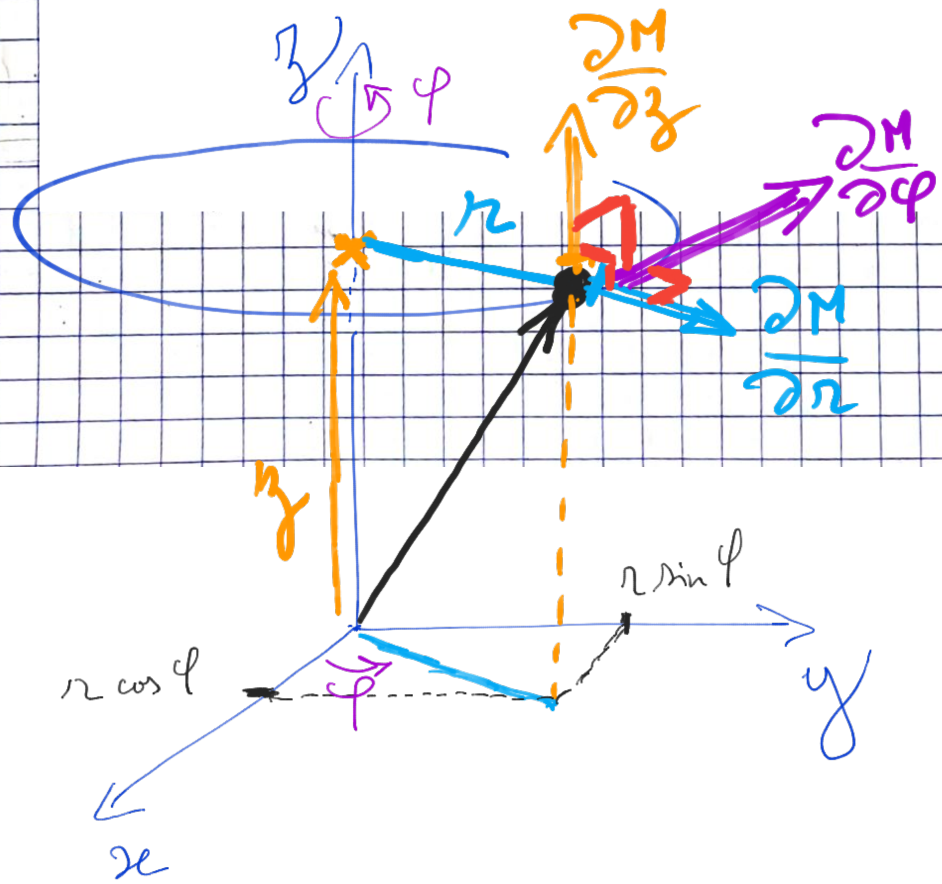
1. Soient $(r,\varphi,z) \in \mathbb{R}_+^* \times \mathbb{R}^2$

$$\boxed{J_f(r,\varphi,z) = \begin{array}{ccc} \multicolumn{3}{c}{\begin{pmatrix} \cos\varphi & -r\sin\varphi & 0 \\ \sin\varphi & r\cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}} \\ \uparrow & \uparrow & \uparrow \\ \frac{\partial M}{\partial r} & \frac{\partial M}{\partial \varphi} & \frac{\partial M}{\partial z} \end{array}}$$

Voir la remarque 5.2 du chap XIV

2. où $M : (r,\varphi,z) \longmapsto (x,y,z) = (r\cos\varphi, r\sin\varphi, z)$.

Voir l'exemple 23 du chapitre XIV



3. Soit $g : \mathbb{R}_+^* \times \mathbb{R}^2 \longrightarrow \mathbb{R}$

Soit $f : \mathbb{R}^3 \to \mathbb{R}$ telle que $g = f \circ M$

On suppose M, f et g différentiable. Alors, d'après la règle de la chaîne :

$$\begin{cases} \dfrac{\partial g}{\partial r} = \dfrac{\partial x}{\partial r}\dfrac{\partial f}{\partial x} + \dfrac{\partial y}{\partial r}\dfrac{\partial f}{\partial y} + \dfrac{\partial z}{\partial r}\dfrac{\partial f}{\partial z} \\ \dfrac{\partial g}{\partial \varphi} = \dfrac{\partial x}{\partial \varphi}\dfrac{\partial f}{\partial x} + \dfrac{\partial y}{\partial \varphi}\dfrac{\partial f}{\partial y} + \dfrac{\partial z}{\partial \varphi}\dfrac{\partial f}{\partial z} \\ \dfrac{\partial g}{\partial z} = \dfrac{\partial x}{\partial z}\dfrac{\partial f}{\partial x} + \dfrac{\partial y}{\partial z}\dfrac{\partial f}{\partial y} + \dfrac{\partial z}{\partial z}\dfrac{\partial f}{\partial z} \end{cases}$$

d'où $\begin{cases} \dfrac{\partial g}{\partial r} = \cos\varphi \dfrac{\partial f}{\partial x} + \sin\varphi \dfrac{\partial f}{\partial y} \\ \dfrac{\partial g}{\partial \varphi} = -r\sin\varphi \dfrac{\partial f}{\partial x} + r\cos\varphi \dfrac{\partial f}{\partial y} \\ \dfrac{\partial g}{\partial z} = \dfrac{\partial f}{\partial z} \end{cases}$

d'où $\begin{cases} \dfrac{\partial g}{\partial r} = \cos\varphi \dfrac{\partial f}{\partial x} + \sin\varphi \dfrac{\partial f}{\partial y} \\ \cos\varphi \dfrac{\partial f}{\partial y} = \dfrac{1}{r}\dfrac{\partial g}{\partial \varphi} + \sin\varphi \dfrac{\partial f}{\partial x} \\ \dfrac{\partial g}{\partial z} = \dfrac{\partial f}{\partial z} \end{cases}$ car $r \neq 0$

d'où $\begin{cases} \dfrac{\partial g}{\partial z} = \dfrac{\partial f}{\partial z} \\ \dfrac{\partial f}{\partial x} = \dfrac{\partial g}{\partial r}\cos\varphi - \dfrac{1}{r}\sin\varphi \dfrac{\partial g}{\partial \varphi} \\ \dfrac{\partial f}{\partial y} = \dfrac{\partial g}{\partial r}\sin\varphi + \dfrac{\cos\varphi}{r}\dfrac{\partial g}{\partial \varphi} \end{cases}$

$$\begin{tikzcd} (r,\varphi,z) \arrow[r, "M"] \arrow[rr, bend right, "g"'] & (x,y,z) \arrow[r, "f"] & f(x,y,z) = g(r,\varphi,z) \end{tikzcd}$$

d'où $\boxed{\vec{\nabla}_{cyl} f(x,y,z) = \dfrac{\partial g}{\partial r}\begin{pmatrix}\cos\varphi \\ \sin\varphi \\ 0\end{pmatrix} + \dfrac{1}{r}\dfrac{\partial g}{\partial \varphi}\begin{pmatrix}-\sin\varphi \\ \cos\varphi \\ 0\end{pmatrix} + \dfrac{\partial g}{\partial z}\begin{pmatrix}0 \\ 0 \\ 1\end{pmatrix}}$