Merci à Gabrielle Tolub.

TD 15 - Exercice n°3 :

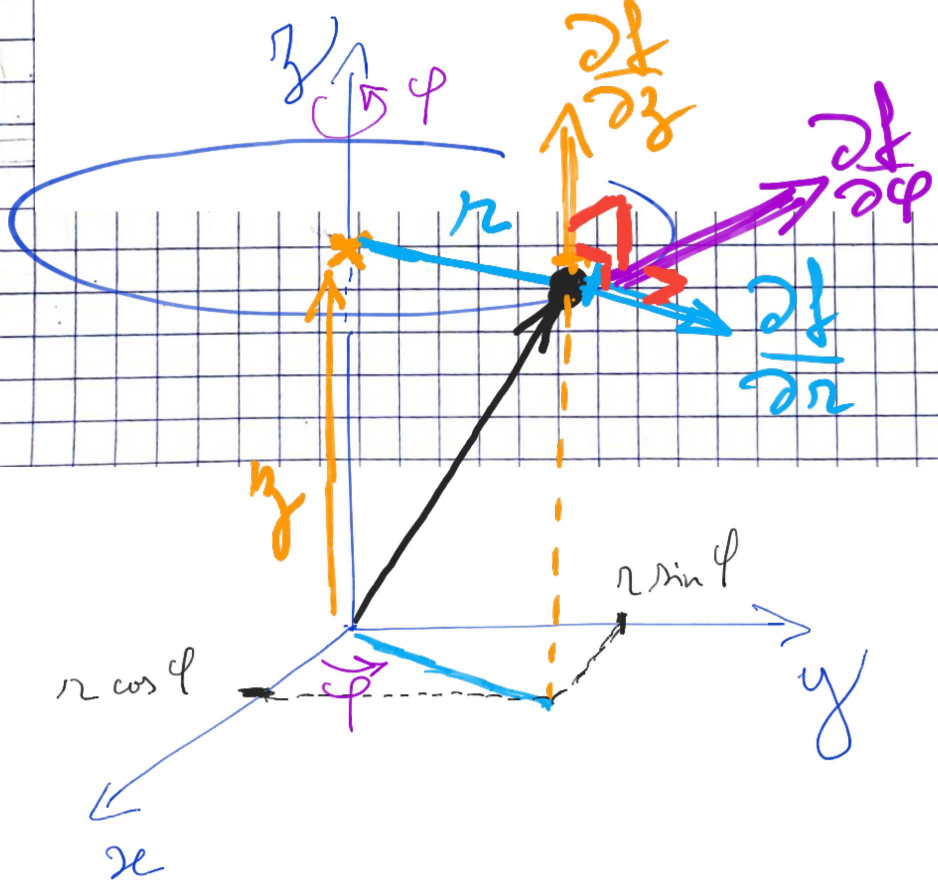
1. Soit $(r, \varphi, z) \in \mathbb{R}_+^* \times \mathbb{R}^2$

$$\boxed{J_f(r,\varphi,z) = \begin{array}{c} \begin{pmatrix} \cos\varphi & -r\sin\varphi & 0 \\ \sin\varphi & r\cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ \begin{array}{ccc} \frac{\partial f}{\partial r} & \frac{\partial f}{\partial \varphi} & \frac{\partial f}{\partial z} \end{array} \end{array}}$$

Voir la remarque 5.2 du chap XIV

2. où $f : (r, \varphi, z) \longmapsto (x, y, z) = (r\cos\varphi, r\sin\varphi, z)$.

Voir l'exemple 2.3 du chapitre XIV



3. Soit $g : \mathbb{R}_+^* \times \mathbb{R}^2 \to \mathbb{R},\ (r,\varphi,z) \mapsto g(r,\varphi,z) = G(x,y,z)$

Soit $G : \mathbb{R}^3 \to \mathbb{R}$ telle que $g = (G \circ f)$

On suppose G et f différentiables. Alors, d'après la règle de la chaîne, g est différentiable et

$$\begin{cases} \dfrac{\partial g}{\partial r} = \dfrac{\partial x}{\partial r}\dfrac{\partial G}{\partial x} + \dfrac{\partial y}{\partial r}\dfrac{\partial G}{\partial y} + \dfrac{\partial z}{\partial r}\dfrac{\partial G}{\partial z} \\ \dfrac{\partial g}{\partial \varphi} = \dfrac{\partial x}{\partial \varphi}\dfrac{\partial G}{\partial x} + \dfrac{\partial y}{\partial \varphi}\dfrac{\partial G}{\partial y} + \dfrac{\partial z}{\partial \varphi}\dfrac{\partial G}{\partial z} \\ \dfrac{\partial g}{\partial z} = \dfrac{\partial x}{\partial z}\dfrac{\partial G}{\partial x} + \dfrac{\partial y}{\partial z}\dfrac{\partial G}{\partial y} + \dfrac{\partial z}{\partial z}\dfrac{\partial G}{\partial z} \end{cases}$$

d'où
$$\begin{cases} \dfrac{\partial g}{\partial r} = \cos\varphi \dfrac{\partial G}{\partial x} + \sin\varphi \dfrac{\partial G}{\partial y} \\ \dfrac{\partial g}{\partial \varphi} = -r\sin\varphi \dfrac{\partial G}{\partial x} + r\cos\varphi \dfrac{\partial G}{\partial y} \\ \dfrac{\partial g}{\partial z} = \dfrac{\partial G}{\partial z} \end{cases}$$

d'où
$$\begin{cases} \dfrac{\partial g}{\partial r} = \cos\varphi \dfrac{\partial G}{\partial x} + \sin\varphi \dfrac{\partial G}{\partial y} \\ \cos\varphi \dfrac{\partial G}{\partial y} = \dfrac{1}{r}\dfrac{\partial g}{\partial \varphi} + \sin\varphi \dfrac{\partial G}{\partial x} \\ \dfrac{\partial g}{\partial z} = \dfrac{\partial G}{\partial z} \end{cases} \quad \text{car } r \neq 0$$

d'où
$$\begin{cases} \dfrac{\partial g}{\partial z} = \dfrac{\partial G}{\partial z} \\ \dfrac{\partial G}{\partial x} = \dfrac{\partial g}{\partial r}\cos\varphi - \dfrac{1}{r}\sin\varphi \dfrac{\partial g}{\partial \varphi} \\ \dfrac{\partial G}{\partial y} = \dfrac{\partial g}{\partial r}\sin\varphi + \dfrac{\cos\varphi}{r}\dfrac{\partial g}{\partial \varphi} \end{cases}$$

$(r,\varphi,z) \xrightarrow{f} (x,y,z) \xrightarrow{G} G(x,y,z)$

$g = G \circ f$

$g(r,\varphi,z)$

d'où
$$\boxed{\nabla G(x,y,z) = \frac{\partial g}{\partial r}\begin{pmatrix}\cos\varphi \\ \sin\varphi \\ 0\end{pmatrix} + \frac{1}{r}\frac{\partial g}{\partial \varphi}\begin{pmatrix}-\sin\varphi \\ \cos\varphi \\ 0\end{pmatrix} + \frac{\partial g}{\partial z}\begin{pmatrix}0 \\ 0 \\ 1\end{pmatrix}}$$