VII- ANNEXE : FORMULAIRE MATHÉMATIQUE

VII-1- Coordonnées cartésiennes

$$\overrightarrow{\text{grad}}\,U = \vec{\nabla}U = \frac{\partial U}{\partial x}\vec{e_x} + \frac{\partial U}{\partial y}\vec{e_y} + \frac{\partial U}{\partial z}\vec{e_z}$$

$$\text{div}\,\vec{A} = \vec{\nabla}.\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\overrightarrow{\text{rot}}\,\vec{A} = \vec{\nabla} \wedge \vec{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\vec{e_x} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\vec{e_y} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\vec{e_z}$$

$$\Delta U = \nabla^2 U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}$$

$$\Delta\vec{A} = \Delta A_x\,\vec{e_x} + \Delta A_y\,\vec{e_y} + \Delta A_z\,\vec{e_z}$$

VII-2- Coordonnées cylindriques

$$\overrightarrow{\text{grad}}\,U = \frac{\partial U}{\partial r}\vec{e_r} + \frac{1}{r}\frac{\partial U}{\partial \theta}\vec{e_\theta} + \frac{\partial U}{\partial z}\vec{e_z}$$

$$\text{div}\,\vec{A} = \frac{1}{r}\frac{\partial (r A_r)}{\partial r} + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z}$$

$$\overrightarrow{\text{rot}}\,\vec{A} = \left(\frac{1}{r}\frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z}\right)\vec{e_r} + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\vec{e_\theta} + \frac{1}{r}\left(\frac{\partial (r A_\theta)}{\partial r} - \frac{\partial A_r}{\partial \theta}\right)\vec{e_z}$$

$$\Delta U = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2} + \frac{\partial^2 U}{\partial z^2}$$

VII-3- Coordonnées sphériques

$$\overrightarrow{\text{grad}}\,U = \frac{\partial U}{\partial r}\vec{e_r} + \frac{1}{r}\frac{\partial U}{\partial \theta}\vec{e_\theta} + \frac{1}{r\sin\theta}\frac{\partial U}{\partial \varphi}\vec{e_\varphi}$$

$$\text{div}\,\vec{A} = \frac{1}{r^2}\frac{\partial \left(r^2 A_r\right)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial (\sin\theta A_\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial \varphi}$$

$$\overrightarrow{\text{rot}}\,\vec{A} = \left[\frac{1}{r\sin\theta}\left(\frac{\partial (\sin\theta A_\varphi)}{\partial \theta} - \frac{\partial A_\theta}{\partial \varphi}\right)\right]\vec{e_r} + \left[\frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial \varphi} - \frac{\partial (r A_\varphi)}{\partial r}\right)\right]\vec{e_\theta} + \left[\frac{1}{r}\left(\frac{\partial (r A_\theta)}{\partial r} - \frac{\partial A_r}{\partial \theta}\right)\right]\vec{e_\varphi}$$

$$\Delta U = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial U}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial U}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 U}{\partial \varphi^2}$$

VII-4- Relations entre opérateurs

◇ Ces propriétés s'établissent rapidement en coordonnées cartésiennes avec la linéarité des opérateurs. On peut utiliser les expressions des différents opérateurs en fonction de l'opérateur $\vec{\nabla}$ pour les retrouver rapidement :

$$\overrightarrow{\text{grad}}\ (U+V) = \overrightarrow{\text{grad}}\,U + \overrightarrow{\text{grad}}\,V$$
$$\overrightarrow{\text{grad}}\ (UV) = U\overrightarrow{\text{grad}}\,V + V\overrightarrow{\text{grad}}\,U$$
$$\overrightarrow{\text{grad}}\ (\vec{A}.\vec{B}) = \vec{A}\wedge\overrightarrow{\text{rot}}\,\vec{B} + \vec{B}\wedge\overrightarrow{\text{rot}}\,\vec{A} + (\vec{A}.\overrightarrow{\text{grad}}\,)\vec{B} + (\vec{B}.\overrightarrow{\text{grad}}\,)\vec{A}$$

$$\text{div}\,(\vec{A}+\vec{B}) = \text{div}\,\vec{A} + \text{div}\,\vec{B}$$
$$\text{div}\,(U\vec{A}) = U\,\text{div}\,\vec{A} + \vec{A}.\overrightarrow{\text{grad}}\,U$$
$$\text{div}\,(\vec{A}\wedge\vec{B}) = \vec{B}.\overrightarrow{\text{rot}}\,\vec{A} - \vec{A}.\overrightarrow{\text{rot}}\,\vec{B}$$

$$\overrightarrow{\text{rot}}\,(\vec{A}+\vec{B}) = \overrightarrow{\text{rot}}\,\vec{A} + \overrightarrow{\text{rot}}\,\vec{B}$$
$$\overrightarrow{\text{rot}}\,(U\vec{A}) = U\overrightarrow{\text{rot}}\,\vec{A} + \overrightarrow{\text{grad}}\,U\wedge\vec{A}$$
$$\overrightarrow{\text{rot}}\,(\vec{A}\wedge\vec{B}) = \vec{A}\,\text{div}\,\vec{B} - \vec{B}\,\text{div}\,\vec{A} + (\vec{B}.\overrightarrow{\text{grad}}\,)\vec{A} - (\vec{A}.\overrightarrow{\text{grad}}\,)\vec{B}$$

$$\Delta U = \text{div}\left(\overrightarrow{\text{grad}}\,U\right) \qquad \text{définition du laplacien scalaire}$$
$$\Delta\vec{A} = \overrightarrow{\text{grad}}\,(\text{div}\,\vec{A}) - \overrightarrow{\text{rot}}\,(\overrightarrow{\text{rot}}\,\vec{A}) \qquad \text{définition du laplacien vectoriel}$$

$$\text{div}\left(\overrightarrow{\text{rot}}\,\vec{A}\right) = 0$$
$$\overrightarrow{\text{rot}}\left(\overrightarrow{\text{grad}}\,U\right) = 0$$

$$(\vec{A}.\overrightarrow{\text{grad}}\,)\vec{A} = \overrightarrow{\text{grad}}\left(\frac{A^2}{2}\right) + (\overrightarrow{\text{rot}}\,\vec{A})\wedge\vec{A}$$
$$\overrightarrow{\text{rot}}\,(\overrightarrow{\text{rot}}\,\vec{A}) = \overrightarrow{\text{grad}}\,(\text{div}\,\vec{A}) - \Delta\vec{A}$$