Exercice ~~[illegible]~~ $\boxed{241}$:

(1) A inversible ssi $XY \neq 0$ ssi $X \neq 0$ et $Y \neq 0$

$P(A \text{ inv}) = P(X \neq 0 \cap Y \neq 0) = P(X \neq 0)\,P(Y \neq 0)$ car X et Y indépendantes

$= (1 - P(X=0))\,(1 - P(Y=0))$

$= \left(1 - \left(\frac{3}{4}\right)^n\right)\left(1 - \left(\frac{1}{4}\right)^n\right) = \frac{4^n - 3^n}{4^n}\,\frac{4^n - 1}{4^n} = \frac{16^n - 4^n - 12^n - 3^n}{16^n}$

(2) Soit $\lambda \in \mathbb{R}$

λ vp de A ssi $\det\begin{pmatrix} X-\lambda & 0 \\ 2 & Y-\lambda \end{pmatrix} = 0$ ssi $(X-\lambda)(Y-\lambda) = 0$ ssi $X = \lambda$ ou $Y = \lambda$

$Sp(A) = \{X, Y\}$.

$E_X = \{(x,y) \in \mathbb{R}^2 \,/\, 2x + (Y-X)y = 0\} = \{(x,y) \in \mathbb{R}^2 \,/\, 2x + (Y-X)y = 0\}$

$\dim(E_X(A)) = 2 - \text{rg}(A - XI_2) = 2 - \text{rg}\begin{pmatrix} 0 & 0 \\ 2 & Y-X \end{pmatrix} = 2 - 1 = 1$

$\dim(E_Y(A)) = 2 - \text{rg}(A - YI_2) = 2 - \text{rg}\begin{pmatrix} X-Y & 0 \\ 2 & 0 \end{pmatrix} = 2 - 1 = 1$

Donc A est diagonalisable ssi $X \neq Y$.

$P(A \text{ diagonalisable}) = P(X \neq Y) = 1 - P(X = Y)$

or $P(X=Y) = P\left(\bigcup_{k=0}^{n} (X=k, Y=k)\right) = \sum_{k=0}^{n} P(X=k \cap Y=k)$ incompatibilité

$= \sum_{k=0}^{n} P(X=k)\,P(Y=k)$ X et Y indépdtes

$= \sum_{k=0}^{n} \binom{n}{k}^2 \left(\frac{1}{4}\right)^n \left(\frac{3}{4}\right)^n = \left(\frac{1}{4}\right)^n \left(\frac{3}{4}\right)^n \sum_{k=0}^{n} \binom{n}{k}^2$

ex 3 : Soit $i \in \mathbb{N}^*$ $\quad Y_i = X_i - 2X_{i+1} + X_{i+2}$

(1) Soit $n \in \mathbb{N}^*$

$E(T_n) = E\left(\frac{1}{n}\sum_{i=1}^{n} Y_i\right) = \frac{1}{n}\sum_{i=1}^{n} E(Y_i)$ lin esp

$E(Y_i) = E(X_i - 2X_{i+1} + X_{i+2})$

$= E(X_i) - 2E(X_{i+1}) + E(X_{i+2})$

$= p - 2p + p = 0.$

$\boxed{E(T_n) = \frac{0}{n} = 0}$

$V(T_n) \overset{V(aX)=a^2V(X)}{=} \frac{1}{n^2} V\left(\sum_{i=1}^{n} Y_i\right)$

$\left(V(Y_i) = V(X_i) + 4V(X_{i+1}) + V(X_{i+2}) = pq + 4pq + pq = 6pq. \right)$ inutile

indépendance de X_i, X_{i+1} et X_{i+2}

$V(T_n) = \frac{1}{n^2} V(X_1 - 2X_2 + X_3 + X_2 - 2X_3 + X_4 + X_3 - 2X_4 + X_5 + X_4 - 2X_5 + X_6$
$+ \dots + X_{n-2} - 2X_{n-1} + X_n + X_{n-1} - 2X_n + X_{n+1} + X_n - 2X_{n+1} + X_{n+2})$

$= \frac{1}{n^2} V(X_1 - X_2 - X_{n+1} + X_{n+2})$

$= \frac{1}{n^2}\left[V(X_1) + V(X_2) + V(X_{n+1}) + V(X_{n+2})\right]$

$= \frac{1}{n^2}[4pq].$