Ex 11
(REF 29) ordre de multiplicité de 1 comme racine de

$$P(X) = X^{2n+1} - (2n+1)X^{n+1} + (2n+1)X^{n} - 1.$$

$n=0$ $\quad P(X) = X - X + X^{0} - 1 = 0.$

$n \geqslant 1.$ $\quad P(1) = 0$

$$P'(X) = (2n+1)X^{2n} - (2n+1)(n+1)X^{n} + n(2n+1)X^{n-1}$$

$P'(1) = 0.$

• si $n=1$ $\quad P'(X) = 3X^2 - 6X + 3$

$P''(X) = 6X - 6$

$P''(1) = 0$

$P'''(X) = 6$ $\quad$ donc $P'''(1) = 6 \neq 0$

dans ce cas 1 est racine triple de $P(X)$.

• si $n \geqslant 2$ $\quad P''(X) = 2n(2n+1)X^{2n-1} - (2n+1)(n+1)nX^{n-1} + n(2n+1)(n-1)X^{n-2}$

$P''(1) = 0.$

• si $n=2$ $\quad P''(X) = 20X^3 - 30X + 10$

$P^{(3)}(X) = 60X^2 - 30$

$P^{(3)}(1) = 30 \neq 0$

1 est racine triple.

• si $n \geqslant 3$ $\quad P^{(3)}(X) = 2n(2n+1)(2n-1)X^{2n-2} - (2n+1)(n+1)(n-1)nX^{n-2}$
$+ n(2n+1)(n-1)(n-2)X^{n-3}$

dès que $n \geqslant 1$, 1 est racine triple de $P(X)$.

$$\begin{aligned}
P^{3}(1) &= 2n(2n+1)(2n-1) - (2n+1)(n+1)(n-1)n + n(2n+1)(n-1)(n-2) \\
&= (2n+1)\left[2n(2n-1) - (n+1)(n-1)n + n(n-1)(n-2)\right] \\
&= (2n+1)\left[2n(2n-1) - n(n-1)(n+1-n+2)\right] \\
&= (2n+1)\left[2n(2n-1) - 3n(n-1)\right] \\
&= (2n+1)\left[4n^2 - 2n - 3n^2 + 3n\right] \\
&= (2n+1)\left[n^2 + n\right] \\
&= (2n+1)(n+1)n \neq 0.
\end{aligned}$$