Exo 4 Réfce 263

(1) $E = \mathbb{R}^n$ $\quad u = \left(\frac{1}{\sqrt{a_1}}, \dots, \frac{1}{\sqrt{a_n}}\right)$ et $v = (\sqrt{a_1}, \dots, \sqrt{a_n})$.

Par l'inégalité de Cauchy-Schwarz : $|u \cdot v| \leq \|u\| \|v\|$

on a donc $\left| \frac{\sqrt{a_1}}{\sqrt{a_1}} + \dots + \frac{\sqrt{a_n}}{\sqrt{a_n}} \right| = |1 + \dots + 1| = n \leq \sqrt{\frac{1}{a_1} + \dots + \frac{1}{a_n}} \times \sqrt{a_1 + \dots + a_n}$

or $a_1 + \dots + a_n = 1$ donc $n \leq \sqrt{\sum_{i=1}^{n} \frac{1}{a_i}}$ donc $\sum_{i=1}^{n} \frac{1}{a_i} \geq n^2$.

(2) $E = \mathbb{R}^n$ $\quad u = (x_1, \dots, x_n)$ et $v = (1, \dots, 1)$

Par l'inégalité de Cauchy-Schwarz : $|u \cdot v| \leq \|u\| \|v\|$

donc $|x_1 + \dots + x_n| \leq \sqrt{x_1^2 + \dots + x_n^2} \sqrt{1^2 + \dots + 1^2}$

donc $|x_1 + \dots + x_n| \leq \sqrt{n} \sqrt{x_1^2 + \dots + x_n^2}$ donc $(x_1 + \dots + x_n)^2 \leq n(x_1^2 + \dots + x_n^2)$

Il y a égalité ssi la famille (u, v) est liée càd si u s'écrit

$k(1, \dots, 1)$, $k \in \mathbb{R}$.

Exercice 6 : refce 266

$u_1 = (1, 2, -1, 1)$

$u_2 = (0, 3, 1, -1)$

$F = \text{Vect}(u_1, u_2)$.

$$u\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \in \left(\text{Vect}(u_1, u_2)\right)^{\perp} \text{ ssi } \begin{cases} u \perp u_1 \\ u \perp u_2 \end{cases} \text{ ssi } \begin{cases} x + 2y - z + t = 0 \\ 3y + z - t = 0 \end{cases}$$

$$\text{ssi } \begin{cases} x = -2y + z - t = \frac{2}{3}z - \frac{2}{3}t + z - t \\ y = -\frac{1}{3}z + \frac{1}{3}t \end{cases}$$

$$\text{ssi } \begin{cases} x = \frac{5}{3}z - \frac{5}{3}t \\ y = -\frac{1}{3}z + \frac{1}{3}t \end{cases}$$

$$= \left(\frac{5}{3}z - \frac{5}{3}t, -\frac{1}{3}z + \frac{1}{3}t, z, t\right)$$

$$F^{\perp} = \text{Vect}\left(\left(\frac{5}{3}, -\frac{1}{3}, 1, 0\right), \left(-\frac{5}{3}, \frac{1}{3}, 0, 1\right)\right).$$

$$= \text{Vect}\left((5, -1, 3, 0), (-5, 1, 0, 3)\right).$$

$\left(\overbrace{(5, -1, 3, 0)}^{v_1}, \overbrace{(-5, 1, 0, 3)}^{v_2}\right)$ est 1 base de $F^{\perp}$. Orthonormalisons la grâce à Gram-Schmidt

- $\varepsilon_1 = \frac{1}{\|v_1\|} v_1 \qquad \|v_1\| = \sqrt{25 + 1 + 9} = \sqrt{35}$

$$= \frac{1}{\sqrt{35}}(5, -1, 3, 0)$$

- $\varepsilon_2' = \alpha \varepsilon_1 + v_2$

$$\alpha = -(v_2 | \varepsilon_1) = -\frac{1}{\sqrt{35}}\begin{pmatrix} -5 \\ 1 \\ 0 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ -1 \\ 3 \\ 0 \end{pmatrix} = -\frac{1}{\sqrt{35}}(-25 - 1) = \frac{26}{\sqrt{35}}$$

$$\varepsilon_2' = \frac{26}{\sqrt{35}} \cdot \frac{1}{\sqrt{35}}(5, -1, 3, 0) + (-5, 1, 0, 3)$$

$$= \frac{1}{35}\left[(130, -26, 78, 0) + (-175, 35, 0, 105)\right]$$

$$\varepsilon_2' = \frac{1}{35}(-45, 9, 78, 105)$$

$$\|\varepsilon_2'\| = \frac{1}{35}\sqrt{45^2 + 9^2 + 78^2 + 105^2} = \frac{1}{35}\sqrt{19215}$$

Exercice (268) 7

① • $\forall (p,q) \in \mathbb{N}^2$, $\langle f_p, f_q \rangle = \int_{-\pi}^{\pi} (f_p f_q)(t)\,dt = \int_{-\pi}^{\pi} \cos pt \cos qt\,dt$

or $\cos pt \cos qt = \frac{1}{2}\left(\cos(p+q)t + \cos(p-q)t\right)$

$\langle f_p, f_q \rangle = \frac{1}{2}\int_{-\pi}^{\pi} \cos(p+q)t\,dt + \frac{1}{2}\int_{-\pi}^{\pi} \cos(p-q)t\,dt$

si $(p+q=0$ et $p-q=0)$ ssi $p=q=0$ alors $\langle f_0, f_0 \rangle = \langle f_p, f_q \rangle = \pi+\pi = 2\pi$

si $\underbrace{p+q=0 \text{ et } p-q\neq 0}_{\text{impossible car } p,q\in\mathbb{N}}$ alors $\langle f_p, f_q \rangle = \frac{1}{2}\cdot 2\pi + \frac{1}{2}\left[\frac{\sin(p-q)t}{p-q}\right]_{-\pi}^{\pi}$

$= \pi + \frac{1}{2}\frac{1}{p-q}\left(\sin(p-q)\pi - \sin((p-q)(-\pi))\right)$

$= \pi + \frac{1}{2}\frac{1}{p-q}\cdot 2\sin((p-q)\pi)$

$\langle f_p, f_p \rangle = \langle f_p, f_q \rangle = \pi + \frac{1}{p-q}\underbrace{\sin((p-q)\pi)}_{=0} = \pi$

si $(p+q\neq 0$ et $p-q=0)$ ssi $p=q$ et $p,q\neq 0$

alors $\langle f_p, f_p \rangle = \frac{1}{2}\int_{-\pi}^{\pi}\cos(2pt)\,dt + \frac{1}{2}2\pi$

$= \frac{1}{2}\left[\frac{\sin(2pt)}{2p}\right]_{-\pi}^{\pi} + \pi$

$= \frac{1}{4p}\left(\sin(2p\pi) - \sin(-2p\pi)\right) + \pi$

$= \frac{1}{4p}\left(2\underbrace{\sin 2p\pi}_{=0}\right) + \pi = \pi$

si $(p+q\neq 0$ et $p-q\neq 0)$ ssi $p\neq q$ et $p\neq 0$ et $q\neq 0$

alors $\langle f_p, f_q \rangle = \frac{1}{2}\left[\frac{\sin((p+q)t)}{p+q}\right]_{-\pi}^{\pi} + \frac{1}{2}\left[\frac{\sin((p-q)t)}{p-q}\right]_{-\pi}^{\pi}$

$= \frac{1}{2}\left(\sin((p+q)\pi) - \sin(-(p+q)\pi)\right) + \frac{1}{2}\left(\sin((p-q)\pi) - \sin((p-q)(-\pi))\right)$

$= \frac{1}{2}\left(2\underbrace{\sin(p+q)\pi}_{=0} + 2\underbrace{\sin(p-q)\pi}_{=0}\right) = 0$

Concl :
$$\begin{cases} \langle f_0, f_0 \rangle = 2\pi \\ \forall p\in\mathbb{N}^* \quad \langle f_p, f_p \rangle = \pi \\ \forall p,q\in\mathbb{N}^*, \text{ avec } p\neq q, \quad \langle f_p, f_q \rangle = 0 \end{cases}$$

• $\forall (p,q)\in\mathbb{N}^2$, $\langle g_p, g_q \rangle = \int_{-\pi}^{\pi} (g_p g_q)(t)\,dt = \int_{-\pi}^{\pi} \sin pt \sin qt\,dt$

or $\sin pt \sin qt = \frac{1}{2}\left(\cos(p-q)t - \cos(p+q)t\right)$

$\langle g_p, g_q \rangle = \frac{1}{2}\int_{-\pi}^{\pi}\cos(p-q)t\,dt - \frac{1}{2}\int_{-\pi}^{\pi}\cos(p+q)t\,dt.$

si $(p+q=0$ et $p-q=0)$ ssi $p=q=0$ alors $\langle g_0, g_0 \rangle = \pi - \pi = 0$

si $p+q=0$ et $p-q\neq 0$: Impossible car $p,q\in\mathbb{N}$

si $(p+q\neq 0$ et $p-q=0)$ ssi $p=q$ et $p\neq 0$ et $q\neq 0$

$$\langle g_p,g_p\rangle = \pi - \frac{1}{2}\int_{-\pi}^{\pi}\cos 2pt\,dt = \pi$$

si $(p+q\neq 0$ et $p-q\neq 0)$ ssi $p\neq q$ et $p\neq 0$ et $q\neq 0$

$$\langle g_p,g_q\rangle = \frac{1}{2}\left[\frac{\sin(p-q)t}{p-q}\right]_{-\pi}^{\pi} - \frac{1}{2}\left[\frac{\sin(p+q)t}{p+q}\right]_{-\pi}^{\pi}$$
$$= 0-0=0$$

Concl : $\begin{cases}\langle g_p,g_p\rangle=\pi & \text{si } p\in\mathbb{N}^*\\ \langle g_0,g_0\rangle=0 \\ \forall p,q\in\mathbb{N}^*,\ p\neq q & \langle g_p,g_q\rangle=0\end{cases}$

- $\forall p,q\in\mathbb{N}$ $\langle f_p,g_q\rangle = \int_{-\pi}^{\pi}(f_pg_q)(t)\,dt = \int_{-\pi}^{\pi}\cos pt\,\sin qt\,dt$

or $\cos pt\sin qt = \frac{1}{2}\left(\sin(p+q)t - \sin(p-q)t\right)$

d'où $\langle f_p,g_q\rangle = \frac{1}{2}\int_{-\pi}^{\pi}\sin(p+q)t\,dt - \frac{1}{2}\int_{-\pi}^{\pi}\sin(p-q)t\,dt$

si $(p+q=0$ et $p-q=0)$ ssi $p=q=0$

$$\langle f_0,g_0\rangle = 0.$$

si $p+q=0$ et $p-q\neq 0$: impossible car $p,q\in\mathbb{N}$

si $(p+q\neq 0$ et $p-q=0)$ ssi $p=q$ et $p\neq 0$ et $q\neq 0$

$$\langle f_p,g_p\rangle = \frac{1}{2}\int_{-\pi}^{\pi}\sin(2pt)\,dt$$
$$=\frac{1}{2}\left[-\frac{\cos 2pt}{2p}\right]_{-\pi}^{\pi}$$
$$=\frac{1}{2}\cdot\frac{1}{2p}\left[-\cos 2p\pi + \cos(-2p\pi)\right]$$
$$=\frac{1}{2}\cdot\frac{1}{2p}\cdot 0 = 0$$

Si $(p+q\neq 0$ et $p-q\neq 0)$ ssi $p\neq q$ et $p\neq 0$ et $q\neq 0$

$$\langle f_p,g_q\rangle = \frac{1}{2}\left[-\frac{\cos(p+q)t}{p+q}\right]_{-\pi}^{\pi} - \frac{1}{2}\left[-\frac{\cos(p-q)t}{p-q}\right]_{-\pi}^{\pi}$$
$$=\frac{1}{2}\frac{1}{p+q}\left[-\cos(p+q)\pi + \cos(-(p+q)\pi)\right] - \frac{1}{2}\frac{1}{p-q}\left[-\cos(p-q)\pi + \cos(-(p-q)\pi)\right]$$
$$=\frac{1}{2}\frac{1}{p+q}[0] - \frac{1}{2}\frac{1}{p-q}[0] = 0$$

Concl : $\begin{cases}\langle f_p,g_p\rangle = 0 \text{ si } p\in\mathbb{N}^* \\ \langle f_0,g_0\rangle=0 \\ \forall p,q\in\mathbb{N}^*, p\neq q, \langle f_p,g_q\rangle=0\end{cases}$ soit $\forall p,q\in\mathbb{N}$ $\langle f_p,g_q\rangle=0$

② On renomme $(e_i)_{i\in[\![1,2n+1]\!]}$ la famille en question.
On a montré que $\langle e_i,e_j\rangle=0$ si $i\neq j$ et $\langle e_i,e_i\rangle=1$ sinon
donc cette famille est orthonormale.