Ex5 : Soit $z \notin \mathbb{U}$, $\left|\dfrac{1-z^n}{1-z}\right| = \left|\sum_{k=0}^{n-1} z^k\right| \underset{\uparrow}{\leq} \sum_{k=0}^{n-1} |z^k| = \sum_{k=0}^{n-1} |z|^k = \dfrac{1-|z|^n}{1-|z|}$

inégalité triangulaire

Ex7 : Soit $x \in \mathbb{R}$, $\sin^2 x = 1-\cos^2 x$ donc $\sqrt{\sin^2 x} = \sqrt{1-\cos^2 x}$

donc $|\sin x| = \sqrt{1-\cos^2 x}$

Ainsi $\sin x = \sqrt{1-\cos^2 x}$ ssi $\sin x = |\sin x|$ ssi $\sin x \geq 0$

ssi $x \in \bigcup_{k\in\mathbb{Z}} [2k\pi, \pi + 2k\pi]$

Soit $x \in \mathbb{R} \setminus \{\frac{\pi}{2} + k\pi, k \in \mathbb{Z}\}$

$\cos^2 x = \dfrac{1}{1+\tan^2 x}$ donc $\sqrt{\cos^2 x} = \sqrt{\dfrac{1}{1+\tan^2 x}}$ donc $|\cos x| = \dfrac{1}{\sqrt{1+\tan^2 x}}$

Ainsi $-\cos x = \dfrac{1}{\sqrt{1+\tan^2 x}}$ ssi $-\cos x = |\cos x|$ ssi $\cos x \leq 0$

ssi $x \in \bigcup_{k\in\mathbb{Z}} [\frac{\pi}{2} + 2k\pi, \frac{3\pi}{2} + 2k\pi]$

Ex13 : Soit $\theta \in \mathbb{R}$.

linéarité de la somme

$S_n(\theta) = \sum_{k=1}^{n} \dfrac{\cos(2k\theta)+1}{2} = \dfrac{1}{2}\sum_{k=1}^{n} \cos(2k\theta) + \sum_{k=1}^{n} \dfrac{1}{2}$

formule de Moivre $\quad = \dfrac{1}{2}\mathrm{Re}\left(\sum_{k=1}^{n} (e^{i2\theta})^k\right) + \dfrac{n}{2}$

$= \dfrac{1}{2}\mathrm{Re}\left(\dfrac{e^{2i\theta} - e^{i2\theta(n+1)}}{1-e^{i2\theta}}\right) + \dfrac{n}{2}$ si $e^{2i\theta} \neq 1$

$= n$ si $e^{2i\theta} = 1$

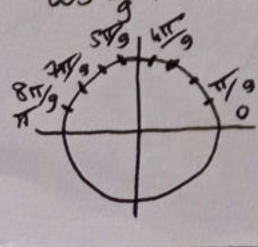
or $\dfrac{e^{2i\theta} - e^{i2\theta(n+1)}}{1-e^{2i\theta}} = \dfrac{e^{i\theta(n+2)}\left(e^{-i\theta n} - e^{i\theta n}\right)}{e^{i\theta}\left(e^{-i\theta} - e^{i\theta}\right)} = e^{i\theta(n+1)} \dfrac{-2i\sin(\theta n)}{-2i\sin(\theta)}$

Donc $\mathrm{Re}\left(\dfrac{e^{2i\theta} - e^{i2\theta(n+1)}}{1-e^{2i\theta}}\right) = \dfrac{\sin(\theta n)}{\sin(\theta)}\cos(\theta(n+1))$

Ccl : si $\theta \notin \{k\pi, k\in\mathbb{Z}\}$, $S_n(\theta) = \dfrac{1}{2}\left[\dfrac{\sin(\theta n)}{\sin(\theta)}\cos(\theta(n+1)) + n\right]$

sinon $S_n(\theta) = n$.

$\cos^2\dfrac{\pi}{9} + \cos^2\dfrac{2\pi}{9} + \cos^2\dfrac{3\pi}{9} + \cos^2\dfrac{4\pi}{9} = S_4\left(\dfrac{\pi}{9}\right) = \dfrac{1}{2}\left[\dfrac{\sin\left(\frac{4\pi}{9}\right)}{\sin\frac{\pi}{9}}\cos\left(\dfrac{\pi}{9}\times 5\right) + 4\right]$



or $\cos\dfrac{5\pi}{9} = -\cos\dfrac{4\pi}{9}$ donc $S_4\left(\dfrac{\pi}{9}\right) = \dfrac{1}{2}\left[\dfrac{-\sin\frac{4\pi}{9}\cos\frac{4\pi}{9}}{\sin\frac{\pi}{9}} + 4\right]$

$S_4\left(\dfrac{\pi}{9}\right) = \dfrac{1}{2}\left[\dfrac{-\frac{1}{2}\sin\frac{8\pi}{9}}{\sin\pi/9} + 4\right] = \dfrac{1}{2}\left[-\dfrac{1}{2}+4\right] = \dfrac{7}{4} \in \mathbb{Q}$ car $\sin\dfrac{8\pi}{9} = \sin\dfrac{\pi}{9}$

Ex 8 $\cos^4 x + \sin^4 x = (\cos^2 x + \sin^2 x)^2 - 2\cos^2 x \sin^2 x = 1 - 2\cos^2 x \sin^2 x$

$$\cos^2 x \sin^2 x = \left(\frac{e^{ix}+e^{-ix}}{2}\right)^2 \left(\frac{e^{ix}-e^{-ix}}{2i}\right)^2$$

$$= -\frac{1}{16}\left[e^{2ix}+2+e^{-2ix}\right]\left[e^{2ix}-2+e^{-2ix}\right]$$

$$= -\frac{1}{16}\left[e^{4ix} \cancel{-2e^{2ix}} + 1 + \cancel{2e^{2ix}} - 4 + \cancel{2e^{-2ix}} + 1 \cancel{-2e^{-2ix}} + e^{-4ix}\right]$$

$$= -\frac{1}{16}\left[e^{4ix}+e^{-4ix}-2\right] = -\frac{1}{16}\left[2\cos(4x)-2\right]$$

$$= -\frac{1}{8}\cos(4x) + \frac{1}{8}$$

donc $\cos^4 x + \sin^4 x = 1 - 2\left(-\frac{1}{8}\cos(4x)+\frac{1}{8}\right)$

$$= 1 + \frac{1}{4}\cos(4x) - \frac{1}{4} = \frac{3}{4} + \frac{1}{4}\cos(4x)$$

donc $\boxed{\cos^4 x + \sin^4 x = \frac{3}{4} + \frac{1}{4}\cos(4x)}$

Ex 9

$$\cos^4 x \times \sin^2 x = \left(\frac{e^{ix}+e^{-ix}}{2}\right)^4 \times \left(\frac{e^{ix}-e^{-ix}}{2i}\right)^2$$

$$= -\frac{1}{2^6}\left(e^{4ix}+4e^{2ix}+6+4e^{-2ix}+e^{-4ix}\right)\left(e^{2ix}-2+e^{-2ix}\right)$$

$$= \frac{-1}{2^6}\Big(e^{6ix}-2e^{4ix}+e^{2ix}+4e^{4ix}-8e^{2ix}+4+6e^{2ix}-12+6e^{-2ix}$$
$$+4-8e^{-2ix}+4e^{-4ix}+e^{-2ix}-2e^{-4ix}+e^{-6ix}\Big)$$

$$= -\frac{1}{2^6}\left(e^{6ix}+2e^{4ix}-e^{2ix}-4-e^{-2ix}+2e^{-4ix}+e^{-6ix}\right) \qquad 2^4=16$$

$$= -\frac{1}{2^6}\left(2\cos 6x+4\cos 4x-2\cos 2x-4\right)$$

$$= -\frac{1}{32}\cos 6x-\frac{1}{16}\cos 4x+\frac{1}{32}\cos 2x+\frac{1}{16}.$$

1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 .

$$\sin^6 x = \left(\frac{e^{ix}-e^{-ix}}{2i}\right)^6 = -\frac{1}{2^6}\left(e^{6ix}-6e^{4ix}+15e^{2ix}-20+15e^{-2ix}-6e^{-4ix}+e^{-6ix}\right)$$

$$= -\frac{1}{2^6}\left(e^{6ix}+e^{-6ix}-6\left(e^{4ix}+e^{-4ix}\right)+15\left(e^{2ix}+e^{-2ix}\right)-20\right)$$

$$= -\frac{1}{2^6}\left(2\cos(6x)-12\cos(4x)+30\cos(2x)-20\right)$$

$$= -\frac{1}{32}\cos(6x)+\frac{3}{16}\cos(4x)-\frac{15}{32}\cos(2x)+\frac{5}{16}.$$

Ex 10

$$\cos(2x) = 2\cos^2 x - 1$$

$$\cos(3x) = \mathrm{Re}\left(\cos(3x)+i\sin(3x)\right) \overset{\text{Moivre}}{=} \mathrm{Re}\left(\left(\cos(x)+i\sin(x)\right)^3\right)$$

$$= \mathrm{Re}\left(\cos^3 x + 3i\cos^2 x\sin x - 3\cos x\sin^2 x - i\sin^3 x\right)$$

$$= \cos^3 x - 3\cos x(\sin^2 x) = \cos^3 x - 3\cos x(1-\cos^2 x)$$

$$= \cos^3 x - 3\cos x + 3\cos^3 x = 4\cos^3 x - 3\cos x.$$

cours

$$\cos(4x) = \mathrm{Re}\left(\cos(4x)+i\sin(4x)\right) \overset{\text{Moivre}}{=} \mathrm{Re}\left(\left(\cos(x)+i\sin(x)\right)^4\right)$$

$$= \mathrm{Re}\left(\cos^4 x + 4i\cos^3 x\sin x - 6\cos^2 x\sin^2 x - 4i\cos x\sin^3 x + \sin^4 x\right)$$

$$= \cos^4 x - 6\cos^2 x\sin^2 x + \sin^4 x$$

$$= \cos^4 x - 6\cos^2 x(1-\cos^2 x) + (1-\cos^2 x)^2$$

$$= \cos^4 x - 6\cos^2 x + 6\cos^4 x + (1+\cos^4 x - 2\cos^2 x)$$

$$= 8\cos^4 x - 8\cos^2 x + 1$$

$$\cos(5x) = \mathrm{Re}\left(\cos(5x)+i\sin(5x)\right) \overset{\text{Moivre}}{=} \mathrm{Re}\left((\cos x+i\sin x)^5\right)$$

$$= \mathrm{Re}\left(\cos^5 x + 5i\cos^4 x\sin x - 10\cos^3 x\sin^2 x - 10i\cos^2 x\sin^3 x + 5\cos x\sin^4 x + i\sin^5 x\right)$$

$$= \cos^5 x - 10\cos^3 x\sin^2 x + 5\cos x\sin^4 x$$

$$= \cos^5 x - 10\cos^3 x(1-\cos^2 x) + 5\cos x(1-\cos^2 x)^2$$

$$= \cos^5 x - 10\cos^3 x + 10\cos^5 x + 5\cos x(1+\cos^4 x - 2\cos^2 x)$$

$$= 11\cos^5 x - 10\cos^3 x + 5\cos x + 5\cos^5 x - 10\cos^3 x$$

$$= 16\cos^5 x - 20\cos^3 x + 5\cos x$$

Ex 6 Soit $M(z)$

1. $|1+z| = a$ ssi $|z-(-1)| = a$ ssi $M \in \mathcal{C}(\Omega, a)$ où $\Omega(-1)$ et $a \geq 0$

2. $0 \leq a \leq |1-z| \leq b$ ssi $0 \leq a \leq |z-1| \leq b$ ssi M appartient à la couronne centrée en $\Omega(1)$ et de rayon intérieur a et de rayon extérieur b.

3. $|z| = |1-z| = 1$ ssi $\begin{cases} |z|=1 \\ |z-1|=1 \end{cases}$ ssi $M \in \mathcal{C}(0,1) \cap \mathcal{C}(\Omega,1)$ où $\Omega(1)$.

Déterminons les points d'intersection de $\mathcal{C}(0,1) \cap \mathcal{C}(\Omega,1)$.

Soit $M(x,y)$ alors $M \in \mathcal{C}(0,1)$ ssi $x^2+y^2=1$
$M \in \mathcal{C}(\Omega,1)$ ssi $(x-1)^2+y^2=1$

Donc $M \in \mathcal{C}(0,1) \cap \mathcal{C}(\Omega,1)$ ssi $\begin{cases} x^2+y^2=1 \\ (x-1)^2+y^2=1 \end{cases}$

ssi $\begin{cases} x^2+y^2=1 \\ x^2+1-2x+y^2=1 \end{cases}$

ssi $\begin{cases} x^2+y^2=1 \\ 1-2x=0 \end{cases}$

ssi $\begin{cases} y^2=\frac{3}{4} \\ x=\frac{1}{2} \end{cases}$ ssi $x=\frac{1}{2}$ et $y=\frac{\sqrt{3}}{2}$ ou $x=\frac{1}{2}$ et $y=-\frac{\sqrt{3}}{2}$

Donc $M \in \mathcal{C}(0,1) \cap \mathcal{C}(\Omega,1)$ ssi $z \in \left\{\frac{1}{2}+i\frac{\sqrt{3}}{2}, \frac{1}{2}-i\frac{\sqrt{3}}{2}\right\}$

4. $1+|z| \leq a$ ssi $|z| \leq a-1$ ssi $M \in \mathcal{C}(0, a-1)$ où $a-1 \geq 0$