Lycee Charlemagne	MPSI2	Annee 2021/22

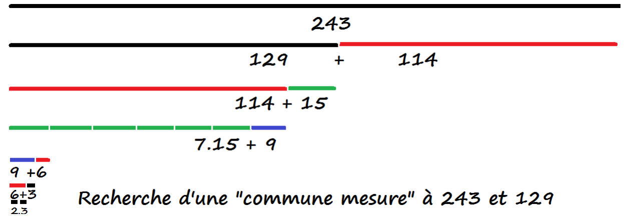
Bézout

et Euclide

Divisions

$$\begin{array}{ccccccc} 243 & = & 1 & \times & 129 & + & 114 \\ & & \swarrow & & & \swarrow & \\ 129 & = & 1 & \times & 114 & + & 15 \\ & & \swarrow & & & \swarrow & \\ 114 & = & 7 & \times & 15 & + & 9 \\ & & \swarrow & & & \swarrow & \\ 15 & = & 1 & \times & 9 & + & 6 \\ 9 & = & 1 & \times & 6 & + & 3 \\ 6 & = & 2 & \times & 3 & & \end{array}$$

3 est le dernier reste non nul.



$$\begin{array}{ccccccc} 243 & - & 1 & \times & 129 & = & 114 \\ 129 & - & 1 & \times & 114 & = & 15 \\ 114 & - & 7 & \times & 15 & = & 9 \\ 15 & - & 1 & \times & 9 & = & 6 \\ 9 & - & 1 & \times & 6 & = & 3 \end{array}$$

$$\begin{array}{rccccc} & 3 & = & 9 & - & 1\times(15-1.9) \\ & 3 & = & 2\times 9 & - & 1\times 15 \\ & 3 & = & 2\times(114-7.15) & - & 1\times 15 \\ \text{puis} & 3 & = & 2\times 114 & - & 15\times 15 \\ & 3 & = & 2\times 114 & - & 15\times(129-1.114) \\ & 3 & = & 17\times(243-129) & - & 15\times 129 \\ & 3 & = & 17\times 243 & - & 32\times 129 \end{array}$$

$243.\mathbb{Z}+129.\mathbb{Z}=3.\mathbb{Z}$

Le plus petit sous-groupe de $(\mathbb{Z},\ +)$ contenant 243 (et ses multiples) et 129 est l'ensemble des multiples de 3.

$243.\mathbb{Z}+129.\mathbb{Z}=\{a.243+b.129 \mid (a,\ b)\in\mathbb{Z}^2\}$

$243.\mathbb{Z}+129.\mathbb{Z}=\{\begin{vmatrix} 243 & u \\ 129 & v \end{vmatrix} \mid (u,\ v)\in\mathbb{Z}^2\}$

Matrices

$$\begin{pmatrix} 129 \\ 114 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}.\begin{pmatrix} 243 \\ 129 \end{pmatrix}$$

$$\begin{pmatrix} 114 \\ 15 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}.\begin{pmatrix} 129 \\ 114 \end{pmatrix}$$

$$\begin{pmatrix} 15 \\ 9 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -7 \end{pmatrix}.\begin{pmatrix} 114 \\ 15 \end{pmatrix}$$

$$\begin{pmatrix} 9 \\ 6 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}.\begin{pmatrix} 15 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} 6 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}.\begin{pmatrix} 9 \\ 6 \end{pmatrix}$$

$$\begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}.\begin{pmatrix} 6 \\ 3 \end{pmatrix}$$

243
129
114
15
9
6
3
0

$$\begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}.\begin{pmatrix} 6 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}.\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}.\begin{pmatrix} 9 \\ 6 \end{pmatrix}$$

$$\begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}.\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}.\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}.\begin{pmatrix} 15 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}.\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}.\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}\ldots\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}.\begin{pmatrix} 243 \\ 129 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}.\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}.\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}.\begin{pmatrix} 0 & 1 \\ 1 & -7 \end{pmatrix}.\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$$

Le produit donne $\begin{pmatrix} 17 & -32 \\ -43 & 81 \end{pmatrix}$

$$\begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 17 & -32 \\ -43 & 81 \end{pmatrix}.\begin{pmatrix} 243 \\ 129 \end{pmatrix}$$

Algorithmes

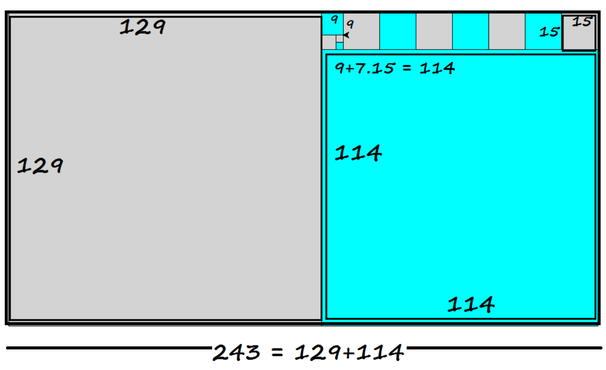
```
def gcd(a, b) : #int, int -> int
....while b != 0 :
........a, b = b, a%b
....return(a)
```

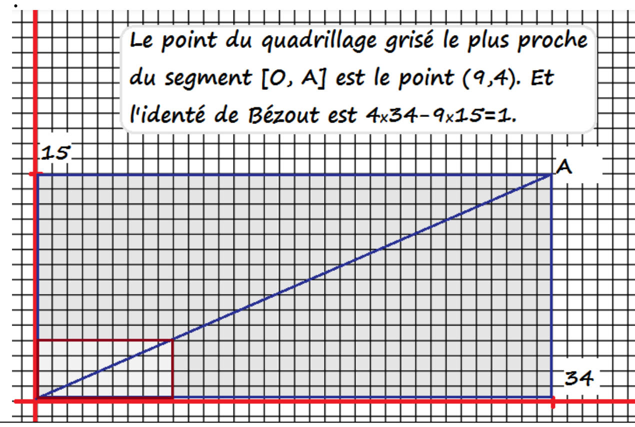
```
def gcdr(a, b) : #int, int -> int
....if b == 0 :
........return(a)
....return(gcdr(b, a%b))
```

En notant justement $\begin{pmatrix} u & v \\ p & q \end{pmatrix}$ les matrices qui interviennent :

```
def Bezout(a, b) :
....u, v = 1, 0
....p, q = 0, 1
....while b != 0 :
........Q = a//b
........a,u,v,b,p,q=b,p,q,a-Q*b,u-Q*p,v-Q*q
....return(r1, u1, u2)
```

Visuels





Fractions

$$\frac{243}{129} = \frac{1 \times 129 + 114}{129} = 1 + \frac{114}{129} = 1 + \frac{1}{\frac{129}{114}}$$

$$\frac{243}{129} = 1 + \frac{1}{\frac{1 \times 114 + 15}{114}} = 1 + \frac{1}{1 + \frac{1}{\frac{115}{15}}}$$

$$\frac{243}{129} = 1 + \frac{1}{1 + \frac{1}{\frac{7 \times 15 + 9}{15}}} = 1 + \frac{1}{1 + \frac{1}{7 + \frac{1}{\frac{15}{9}}}}$$

$$\frac{243}{129} = 1 + \frac{1}{1 + \frac{1}{7 + \frac{1}{1 + \frac{1}{\frac{9}{6}}}}}$$

$$\frac{243}{129} = 1 + \frac{1}{1 + \frac{1}{7 + \frac{1}{1 + \frac{1}{1 + \frac{1}{\frac{6}{3}}}}}}$$

enfin $\boxed{\frac{243}{129} = 1 + \frac{1}{1 + \frac{1}{7 + \frac{1}{1 + \frac{1}{1 + \frac{1}{2}}}}}}$

On efface le dernier entier :

$$1 + \frac{1}{1 + \frac{1}{7 + \frac{1}{1 + \frac{1}{1+0}}}} = \frac{?}{?}$$

$$1 + \frac{1}{1 + \frac{1}{7 + \frac{1}{2}}} = \frac{?}{?}$$

$$1 + \frac{1}{1 + \frac{1}{\frac{15}{2}}} = 1 + \frac{1}{1 + \frac{2}{15}} = \frac{?}{?}$$

$$1 + \frac{1}{\frac{17}{15}} = 1 + \frac{15}{17} = \frac{?}{?}$$

$\frac{32}{17} = \frac{?}{?}$ Etonnant, non ?