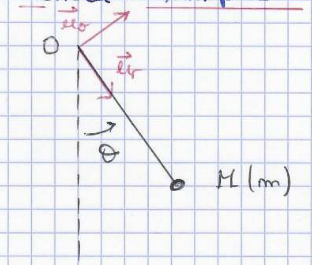
1. Pendule simple – CCINP

Pendule simple



A $t = t_0$ $\quad \theta(t_0) = 0$

$\vec{v}(t_0) = V_0 \vec{u_\theta}$

1) . Oscillations

. Fronde

. Arc de cercle ⊕ chute libre puis oscillations.

2) Σ Ref RdF poids $\vec{P} = mg(\cos\theta \, \vec{e_r} - \sin\theta \, \vec{e_\theta})$

tension $\vec{T} = -T \vec{u_r}$

Mouvement circulaire $\vec{OM} = l \vec{e_r}$

$\vec{v} = l\dot{\theta} \, \vec{e_\theta}$

$\vec{a} = -l\dot{\theta}^2 \vec{u_r} + l\ddot{\theta} \, \vec{e_\theta}$

PFD $\quad m\vec{a} = \vec{P} + \vec{T}$

Selon $\vec{e_r}$ $\quad -m\frac{v^2}{l} = -T + mg\cos\theta$

$$\boxed{T(\theta) = mg\cos\theta + m\frac{v^2(\theta)}{l}}$$

TEC entre $\theta = 0$ et θ

$$\frac{1}{2}mv^2 - \frac{1}{2}mV_0^2 = \int_{\theta=0}^{\theta} -mg\sin\theta \, l d\theta$$

$$= mgl(\cos\theta - 1)$$

$$mv^2(\theta) = mV_0^2 + 2mgl(\cos\theta - 1)$$

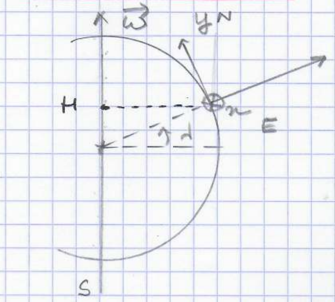
Il vient $\boxed{T(\theta) = m\frac{V_0^2}{l} + 3mg\cos\theta - 2mg.}$

3) Il faut $T(\pi) > 0$

$\Leftrightarrow \quad m\frac{V_0^2}{l} > 5mg \qquad V_0 > \sqrt{5gl}$

2. Déviation vers l'est – CCINP

Déviation vers l'est -



1) Σ

Ref terrestre galiléen

BdF le poids $\vec P = m\vec g$

2e loi de Newton

$$m\vec a = m\vec g$$

$$\vec a = \begin{vmatrix} 0 \\ 0 \\ -g \end{vmatrix} \quad \vec v = \begin{vmatrix} 0 \\ 0 \\ -gt \end{vmatrix} \quad \vec{OM} = \begin{vmatrix} 0 \\ 0 \\ -g\frac{t^2}{2}+h \end{vmatrix}$$

$x(t) = 0$

$z(t) = h - g\frac{t^2}{2}$

$$\boxed{\tau = \sqrt{\frac{2h}{g}}}$$

AN $\tau = \sqrt{\frac{300}{9{,}81}}$ $\quad \tau = 5{,}53\ s$

2) $\vec F_{ie} = -m\vec a_e = -2m\vec\omega\wedge\vec v$

avec $\vec\omega = \omega(\sin\lambda\,\vec e_z + \cos\lambda\,\vec e_y)$

$$\frac{\|\vec F_{ie}\|}{\|\vec P\|} \approx \frac{2m\omega g\tau}{mg} \quad \text{avec } \omega = \frac{2\pi}{T}$$

AN $\frac{\|\vec F_{ie}\|}{\|\vec P\|} = 8\ 10^{-4} \ll 1 \rightarrow$ perturbation.

x est principalement perturbé

$$\vec F_{ie} = -2m\omega\begin{vmatrix} 0 \\ \cos\lambda \\ \sin\lambda \end{vmatrix} \wedge \begin{vmatrix} \dot x \\ \dot y \\ \dot z \end{vmatrix} = 2m\omega\begin{vmatrix} -\dot z\cos\lambda + \cancel{\dot y\sin\lambda} \\ -\dot x\sin\lambda \\ \dot x\cos\lambda \end{vmatrix}$$

3) $m\ddot x = +2m\omega g t\cos\lambda$

$\dot x = 2\omega g\cos\lambda\, t^2$

$x(t) = \omega g\cos\lambda\,\frac{t^3}{3}$ $\quad$ pour $\lambda = 45°$

$X_f = \omega g\cos\lambda\,\frac{2h}{g}\sqrt{\frac{2h}{g}}\cdot\frac{1}{3}$ $\quad X_f = 2{,}8$ cm

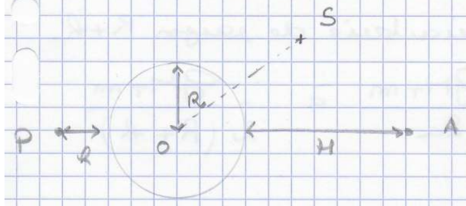
3. Satellite – CCINP

Satellite

Σ = satellite S

Réf géocentrique galiléen

1) Mouvement circulaire



$\vec F = -\frac{M_T m}{r^2}\mathcal{G}\,\vec e_r$

$\vec{OS} = R\vec e_r$

$\vec v = R\dot\theta\,\vec e_\theta = v\,\vec e_\theta$

$\vec a = -\frac{v^2}{R}\vec e_r + \frac{dv}{dt}\vec e_\theta$

PFD $+m\frac{v^2}{R} = +\frac{M_T m\mathcal{G}}{R^2}$

$E_p = -\frac{\mathcal{G}M_T m}{R}$

$E_m = \frac{1}{2}mv^2 - \mathcal{G}\frac{M_T m}{R} = -\frac{1}{2}\frac{\mathcal{G}M_T m}{R}$

On a $E_m = -\frac{\mathcal{G}M_T m}{2a}$

Ellipse : $2a = h + 2R + H \Leftrightarrow$

$$\boxed{H = 2(a-R) - h.}$$

2) T = période de rotation.

Pour circulaire $\dot\theta = \frac{2\pi}{T} = \frac{v}{R}$

$\frac{4\pi^2}{T^2} = \frac{\mathcal{G}M_T}{R^3}$ $\quad \frac{T^2}{R^3} = \frac{4\pi^2}{\mathcal{G}M_T} = $ cte.

$$\boxed{T = \frac{2\pi}{\sqrt{\mathcal{G}M_T}}\,a^{3/2}}$$

3) $\frac{1}{2}\cancel{m}v_p^2 - \frac{\mathcal{G}M_T\cancel{m}}{(R+h)} = -\frac{\mathcal{G}M_T\cancel{m}}{2a}$.

$v_p = \sqrt{2}\,.\sqrt{\mathcal{G}M_T}\sqrt{\frac{1}{R+h} - \frac{1}{2a}}$

$v_A = \sqrt{2\mathcal{G}M_T}\sqrt{\frac{1}{R+H} - \frac{1}{2a}}$

4/ Il faut être en A ou P pour $\vec{OM} \perp$ à $\vec{v}$

Pour P trajectoire circulaire de rayon $R+h$

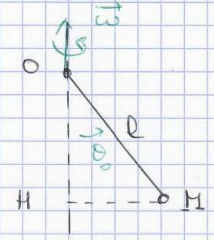
E_m passe de $-\dfrac{\mathcal{G}M_T m}{2a}$ à $-\dfrac{\mathcal{G}M_T m}{2(R+h)}$

$v \nearrow$.

Pour A $v \downarrow$.

4. Pendule conique – CCINP

Pendule conique



Σ = pt matériel M

Réf terrestre galiléen

BdF $\vec{P}$ poids

$\vec{T}$ tension du fil.

Repère H $\vec{e_r}$ $\vec{e_\alpha}$ $\vec{e_z}$

$\vec{HM} = l\sin\theta_0\,\vec{e_r}$ Mvt circulaire unif.

$\vec{v} = l\sin\theta_0\,\omega\,\vec{e_\theta}$

$\vec{a} = -l\sin\theta_0\,\omega^2\,\vec{e_r}$

$\vec{P} = -mg\vec{e_z}$ $\vec{T} = T\cos\theta_0\,\vec{e_z} - T\sin\theta_0\,\vec{e_r}$

PFD $m\vec{a} = \vec{T} + \vec{P}$

$\vec{e_r}$: $-ml\sin\theta_0\,\omega^2 = -T\sin\theta_0$

$\vec{e_z}$: $0 = -mg + T\cos\theta_0 \Leftrightarrow T = \dfrac{mg}{\cos\theta_0}$

Il vient $\cancel{m}l\sin\theta_0\,\omega^2 = \cancel{m}g\dfrac{\sin\theta_0}{\cos\theta_0}$ $\theta_0 = 0$ ou π

$\cos\theta_0 = \dfrac{g}{l\omega^2} < 1$ Il faut $\omega > \sqrt{\dfrac{g}{l}}$

OU

Réf tournant $\rightarrow$ + $\vec{F}_{ient} = +m\omega^2\vec{HM}$

$\vec{F}_{ic} = \vec{0}$ car $\vec{v} = \vec{0}$

PFD $\vec{0} = \vec{P} + \vec{T} + \vec{F}_{ient} + \vec{F}_{ic}$

$\vec{e_z}$ $0 = -mg + T\cos\theta_0$

$\vec{e_r}$ $0 = 0 - T\sin\theta_0 + m\omega^2 l\sin\theta_0$. Idem.

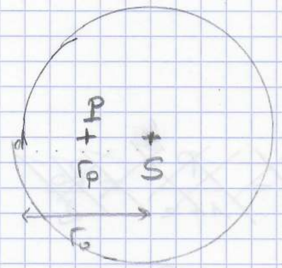
ou étude générale de l'équilibre

$E_p(\theta) = mgl(1-\cos\theta) - \frac{1}{2}ml^2\sin^2\theta\,\omega^2$.

$\frac{dE_p}{d\theta}(\theta) = mgl\sin\theta - \frac{2}{2}ml^2\sin\theta\cos\theta\,\omega^2$

$= ml\sin\theta\,[g - l\omega^2\cos\theta]$

5. Comète solaire



$v_p = 2 v_0$

$r_p = \frac{r_0}{2}$

$$E_m = \frac{1}{2} m (v_0 2)^2 - \frac{\mathcal{G} m M_s}{r_0} 2$$

$$= 2 m v_0^2 - \frac{\mathcal{G} m M_s 2}{r_0}$$

Or pour la Terre

$$E_{m_T} = \frac{1}{2} m_T v_0^2 - \frac{\mathcal{G} M_s m_T}{r_0} = - \frac{\mathcal{G} M_s m_T}{2 r_0}$$

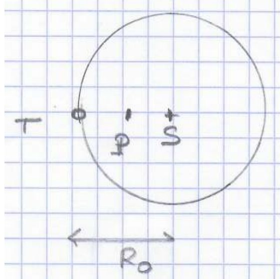
Soit $v_0^2 = \frac{\mathcal{G} M_s}{r_0}$

Il vient $E_m = 0$ Trajectoire parabolique

2) $0 = \frac{1}{2} m v^2(r) - \frac{\mathcal{G} M_s m}{r}$

$$\rightarrow \boxed{v(r) = \sqrt{\frac{2 \mathcal{G} M_s}{r}}}$$

Centrale.



1) PFD $- M_0 \frac{V_0^2}{R_0} = - \frac{\mathcal{G} M_s M_0}{R_0^2}$

$$\rightarrow \boxed{V_0 = \sqrt{\frac{\mathcal{G} M_s}{R_0}}}$$

$$E_c = \frac{1}{2} M_0 V_0^2 = \frac{1}{2} \frac{\mathcal{G} M_s M_0}{R_0}$$

$$\boxed{E_m = - \frac{1}{2} \frac{\mathcal{G} M_0 M_s}{R_0}}$$

$$\vec{L}_S(T) = R_0 \vec{e_r} \wedge V_0 \vec{e_\theta} M_0$$

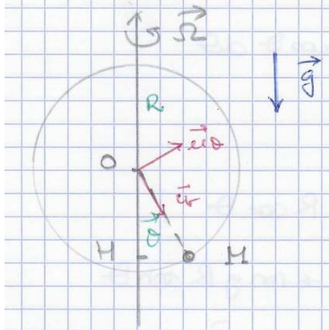
$$= M_0 V_0 R_0 \vec{e_z}$$

2) ... Parabole

$$E_m = \frac{1}{2} m v^2 - \frac{\mathcal{G} M_s m}{r} = 0$$

6. Anneau sur cerceau tournant – Centrale, Navale

Anneau sur cerceau tournant



Σ = pt mat M

Réf de l'anneau non galiléen

BdF

- poids $\vec{P}$

$$\vec{P} = mg (\cos\theta \, \vec{e_r} - \sin\theta \, \vec{e_\theta})$$

- réaction de l'anneau

$$\vec{N} = N_r \vec{u_r} + N_z \vec{u_z}$$

- force d'inertie d'entraînement

$$\vec{f_{ient}} = m \Omega^2 \vec{HM} = m \Omega^2 R \sin\theta \, [\sin\theta \, \vec{e_r} + \cos\theta \, \vec{e_\theta}]$$

- force d'inertie de Coriolis

$$\vec{f_{ic}} = -2m \vec{\Omega} \wedge \vec{v} = \vec{0} \text{ à l'équilibre}$$

PFD à l'équilibre $\vec{0} = \vec{P} + \vec{N} + \vec{f_{ient}} + \vec{f_{ic}}$.

Selon $\vec{e_\theta}$: $0 = -mg \sin\theta + m \Omega^2 R \sin\theta \cos\theta$

$$= m \sin\theta \, [\Omega^2 R \cos\theta - g]$$

- $\sin\theta = 0$ $\quad \theta_{e1} = 0$ $\quad \theta_{e2} = \pi$
- $g = \Omega^2 R \cos\theta$ $\quad \cos\theta_{e3} = \frac{g}{\Omega^2 R}$ si < 1.

$\vec{P}$ force conservative $\delta W(\vec{P}) = \vec{P} \cdot \vec{dOM}$

$$= \vec{P} \cdot R \, d\theta \, \vec{e_\theta}$$

$$- dE_{p_P} = - mg R \sin\theta \, d\theta$$

$$E_{pp}(\theta) = - mg R \cos\theta$$

$\vec{N} \perp$ à l'anneau ne travaille pas

$$\delta W(\vec{f}_{ient}) = \vec{f}_{ient} \cdot d\vec{OM}$$

$$-dE_{p\,ent} = m\Omega^2 R^2 \sin\theta \cos\theta \, d\theta$$

$$E_{p\,ent} = \frac{1}{2} m \Omega^2 R^2 \cos^2\theta$$

$$E_p = \frac{1}{2} m \Omega^2 R^2 \cos^2\theta - mgR\cos\theta$$

$$\frac{dE_p}{d\theta} = -m\Omega^2 R^2 \cos\theta \sin\theta + mgR\sin\theta$$

$$= mR\sin\theta \left[g - R\Omega^2 \cos\theta \right]$$

$$\frac{d^2E_p}{d\theta^2} = mR\cos\theta \left[g - R\Omega^2\cos\theta \right]$$

$$+ mR\sin\theta \; R\Omega^2 \sin\theta$$

Pour $\theta = 0$ $\quad \frac{d^2E_p}{d\theta^2}(\theta_{e_1}) = mR\left[g - R\Omega^2\right]$

si $g > R\Omega^2$ éq stable

si $g < R\Omega^2$ éq instable

Pour $\theta = \pi$ $\quad \frac{d^2E_p}{d\theta^2}(\theta_{e_2}) = -mR\left[g + R\Omega^2\right] < 0$

toujours instable

Pour $\cos\theta_{e_3} = \frac{g}{R\Omega^2}$ quand $g < R\Omega^2$

$$\frac{d^2E_p}{d\theta^2}(\theta_{e_3}) = mR\frac{g}{R\Omega^2}\Big[\underbrace{g - R\Omega^2\frac{g}{R\Omega^2}}_{=0}\Big]$$

$$+ mR^2\Omega^2\left(1 - \frac{g^2}{R^2\Omega^4}\right) > 0$$

Stable.

Période des petites oscillations.

$$\omega = \sqrt{\frac{k}{mR^2}} = \frac{2\pi}{T}$$

avec $E_p = \frac{1}{2} k (\theta - \theta_e)^2$

$$\frac{dE_p}{d\theta} = k\theta = -RF(\theta) \; ; \; \frac{d^2E_p}{d\theta^2} = k.$$

$$\delta W(\vec{F}) = \vec{F} \cdot d\vec{OM} = F(\theta) R\, d\theta = -dE_p.$$

$$\frac{1}{R}\frac{dE_p}{d\theta} = -F(\theta)$$

PFD $\; mR\ddot{\theta} = F(\theta)$, $\quad \ddot{\theta} - F(\theta)/mR = 0 \quad \ddot{\theta} + k/mR^2 \, \theta = 0$

$$T = \frac{2\pi}{\sqrt{k}} \cdot \sqrt{mR^2} = 2\pi \sqrt{\frac{mR^2}{d^2E_p/d\theta^2(\theta_e)}}$$

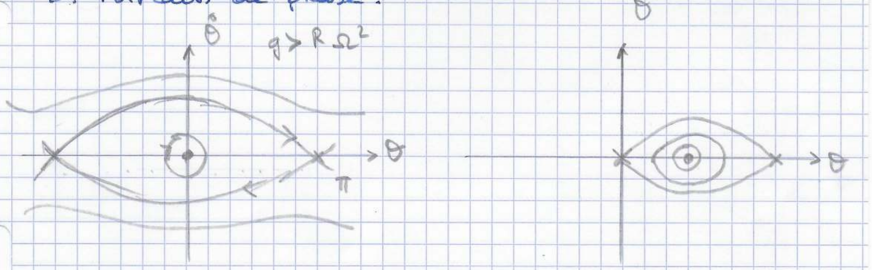
Pour $\theta = 0$ qd $g > R\Omega^2$

$$T_0 = 2\pi \frac{R}{\sqrt{R(g - R\Omega^2)}} = \frac{2\pi}{\sqrt{\frac{g}{R} - \Omega^2}}$$

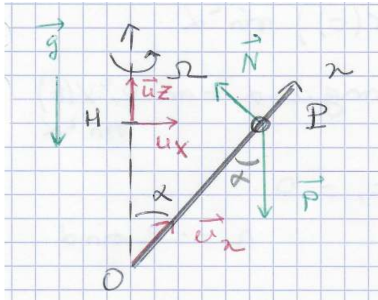
Pour $\cos\theta_{e_3} = g/R\Omega^2$ $\quad g < R\Omega^2$

$$T_3 = 2\pi \frac{\sqrt{mR^2}}{\sqrt{mR^2\Omega^2\left(1 - \frac{g^2}{R^2\Omega^4}\right)}} = \frac{2\pi}{\sqrt{\Omega^2 - g^2/R^2\Omega^2}}$$

3. Portraits de phase.



7. Tige tournante – CCINP, MT



Σ = perle P

Réf de la tige tournant non galiléen

BdF

* le poids $\vec{P} = m\vec{g}$
* la réaction normale du support
* la force d'inertie d'entraînement
* la force d'inertie de Coriolis

1) A l'équilibre $\vec{0} = \vec{P} + \vec{N} + \vec{f_{ient}} + \underbrace{\vec{f_{ic}}}_{=\vec{0} \text{ car } \vec{v}=\vec{0}}$

Selon $\vec{u_x}$ $\quad 0 = P_x + f_{ient\,x}$

avec $\vec{f_{ient}} = m\Omega^2 \vec{HP}$

$= m\Omega^2 x \sin\alpha\, \vec{u_X}$

$\vec{P} = -mg\vec{u_z}$

Avec les énergies potentielles

$E_{Pp}(x) = mg\,x\cos\alpha$

$E_{P\,ient}(x) = -\frac{1}{2}m(x^2\sin^2\alpha)\Omega^2$

$0 = -mg\cos\alpha + m\Omega^2 x_{eq}\sin^2\alpha$

$$\boxed{x_{eq} = \frac{g\cos\alpha}{\Omega^2\sin^2\alpha}}$$

$\frac{dEp}{dx} = 0 = mg\cos\alpha - m x_{eq}\Omega^2\sin^2\alpha$

2) On pose $x(t) = x_{eq} + X(t)$

$v(t) = \dot{X}(t)$

$a(t) = \ddot{X}(t)$

PFD $\quad m\vec{a} = \vec{P} + \vec{N} + \vec{f_{ient}} + \vec{f_{ic}}$

avec $\vec{f_{ic}} = -2m\vec{\Omega}\wedge\vec{v} \perp \grave{a}\ \vec{u_x}$.

PFD selon $\vec{u_x}$: $m\ddot{X} = -mg\cos\alpha + m\Omega^2 (x_{eq} + X(t))\sin^2\alpha$

$\Leftrightarrow m\ddot{X} = -mg\cos\alpha + mg\cos\alpha + m\Omega^2 X(t)\sin^2\alpha$

$\ddot{X} - \Omega^2\sin^2\alpha\, X(t) = 0$.

$r^2 - \Omega^2\sin^2\alpha = 0$. $\quad r = \Omega\sin\alpha$

$X(t) = Ae^{rt} + Be^{-rt}$

Instable.

8. Bille dans bol – Mines

$M(r\ \theta\ z)$

Paraboloïde $r^2 = az$

1) Σ = point matériel M

Réf terrestre galiléen

Bilan des forces

$\vec{P}$ = poids $\quad \vec{P} = -mg\,\vec{e_z}$

$\vec{N}$ = réaction du bol (ne travaille pas)

Cinématique $\quad \overrightarrow{OM} = r\vec{e_r} + z\vec{e_z}$

$$\vec{v} = \dot{r}\,\vec{e_r} + r\dot{\theta}\,\vec{e_\theta} + \dot{z}\,\vec{e_z}$$

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\,\vec{e_r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\,\vec{e_\theta} + \ddot{z}\,\vec{e_z}$$

On a toujours $r^2 = az$

soit $2\dot{r}r = a\dot{z}$

TEM $\quad \Delta E_m = 0$

$$\frac{1}{2}mv^2 + mgz = \text{cte}.$$

TMC par rapport à Oz

$$\frac{dL_z}{dt} = \mathcal{M}_{Oz}(\vec{P}) + \underbrace{\mathcal{M}_{Oz}(\vec{N})}_{=0}$$

$$\mathcal{M}_{Oz}(\vec{P}) = (\overrightarrow{OM} \wedge \vec{P})\cdot\vec{e_z}$$

$$= \left((r\vec{e_r} + z\vec{e_z}) \wedge (-mg\vec{e_z})\right)\cdot\vec{e_z}$$

$$= 0$$

$\vec{N} \perp$ au bol $\quad \vec{N}\cdot d\overrightarrow{OM} = 0$

$$N_r\,dr + r\,d\theta\,N_\theta + N_z\,dz = 0$$

avec $2r\,dr = a\,dz \quad \mathcal{M}_{Oz}(\vec{N}) = 0$

Il y a conservation du moment cinétique par rapport à Oz

$$m r^2 \dot{\theta} = C.$$

2) $E_m = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + \frac{1}{2}m\dot{z}^2 + mgz$

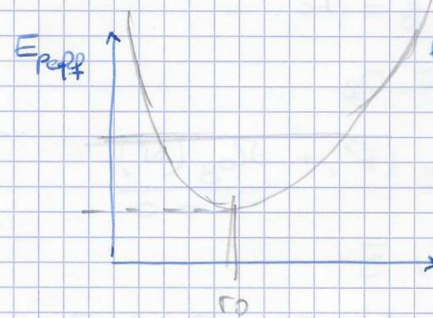
avec $\dot{\theta} = \frac{C}{mr^2}$ et $z = \frac{r^2}{a} \quad \dot{z} = \frac{2r\dot{r}}{a}$

Il vient

$$E_m = \frac{1}{2}m\left[\dot{r}^2 + \frac{C^2}{m^2r^2} + \frac{4r^2\dot{r}^2}{a^2}\right] + mg\frac{r^2}{a}$$

$$= \frac{1}{2}\dot{r}^2\underbrace{\left(m + \frac{4m}{a^2}r^2\right)}_{G(r)} + \underbrace{mg\frac{r^2}{a} + \frac{C^2}{2mr^2}}_{E_{peff}(r)}$$

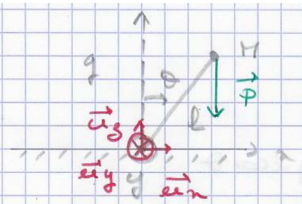
$$\frac{dE_{peff}}{dr}(r) = 2mg\frac{r}{a} - \frac{C^2}{mr^3} \qquad r_0 = \sqrt[4]{\frac{aC^2}{2m^2g}}$$



$$E_{peff}(r_0) = \frac{\cancel{m}g}{a}\sqrt{\frac{aC^2}{\cancel{m^2}g}} + \frac{\cancel{C^2}\sqrt{m^2g}}{2m\sqrt{a}\,\cancel{C^2}}$$

$$= \frac{3}{2}\frac{\sqrt{g}\,C}{\sqrt{a}}$$

9. Couple de rappel – CCINP



Σ = tige + masse.

Réf terrestre galiléen R_g

Bilan des actions

* le poids $\vec{P}$ en M

$\vec{P} = -mg\,\vec{e}_z$

* le couple de rappel $\mathcal{C} = -C\theta$.
* liaison pivot en O.

TMC / Oy fixe dans R_g

$$\frac{dL_{Oy}(S)}{dt} = \mathcal{C} + \mathcal{M}_{Oy}(\vec{P}) + \underbrace{\mathcal{M}_{Oy}(\text{liaison})}_{=0 \text{ pivot parfait}}$$

avec $L_{Oy}(S) = (\vec{OM} \wedge m\vec{v}(M)) \cdot \vec{e}_y$

$= m\ell^2 \dot{\theta}$

or $\mathcal{M}_{Oy}(\vec{P}) = (\vec{OM} \wedge m\vec{g}) \cdot \vec{e}_y$

$= (\ell\vec{e}_r \wedge (-mg\,\vec{e}_z)) \cdot \vec{u}_y$ $\qquad \vec{u}_r = \ell\sin\theta\,\vec{e}_x + \ell\cos\theta\,\vec{e}_z$

$= +mg\ell\sin\theta$

Il vient $m\ell^2\ddot{\theta} = mg\ell\sin\theta - \mathcal{C}\theta$

$$\Leftrightarrow \ddot{\theta} + \frac{\mathcal{C}}{m\ell^2}\theta - \frac{g}{\ell}\sin\theta = 0.$$

Pour θ petit $\sin\theta \simeq \theta$

$$\ddot{\theta} + \underbrace{\left(\frac{\mathcal{C}}{m\ell^2} - \frac{g}{\ell}\right)}_{>0 \text{ pour stable}}\theta = 0$$

$\Rightarrow \mathcal{C} > mg\ell$. $\qquad \omega^2 = \frac{4\pi^2}{T^2} = \left(\frac{\mathcal{C}}{m\ell^2} - \frac{g}{\ell}\right)$

$$T = \frac{2\pi}{\sqrt{\frac{\mathcal{C}}{m\ell^2} - \frac{g}{\ell}}}$$

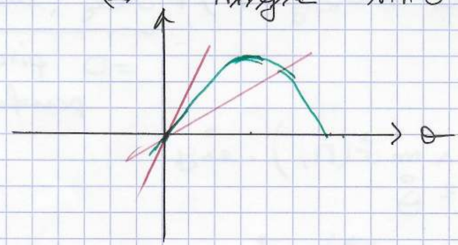
Avec les énergies $E_p = mg\ell\cos\theta$

$E_{p\,\text{couple}} = \frac{1}{2}\mathcal{C}\theta^2$

$\mathcal{P}_{\text{couple}} = -\mathcal{C}\theta\dot{\theta}$

$$\frac{dE_p}{d\theta} = -mg\ell\sin\theta + \mathcal{C}\theta = 0.$$

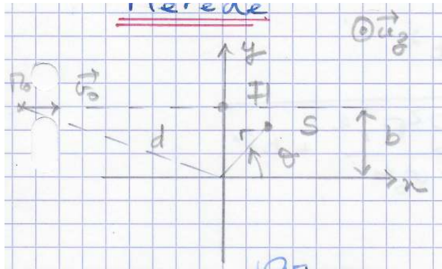
$\Leftrightarrow \cancel{mg\ell}\ \sin\theta = \frac{\mathcal{C}}{mg\ell}\theta$



$$\frac{d^2E_p}{d\theta^2}(\theta) = -mg\ell\cos\theta + \mathcal{C}$$

pour $\theta = 0 \longrightarrow \mathcal{C} > mg\ell$.

10. Météore – CCINP



1) $\vec{\sigma}_O(M)$ et E_m sont conservées.

$\left(\vec{F} = -\frac{\mathcal{G}M_T m}{r^2}\vec{u}_r = \right.$ force centrale conservative$\left.\right)$.

$$E_p = -\frac{\mathcal{G}M_T m}{r}$$

2) a) $\vec{\sigma}_O(M_\infty) = \vec{OM}_\infty \wedge m\vec{v}_0$ par définition

$= (\vec{OH} + \vec{HM}_\infty) \wedge m\vec{v}_0$

$= b\vec{u}_y \wedge m v_0 \vec{u}_x = -m b v_0 \vec{u}_z$.

$$\boxed{\vec{\sigma}_O(M_\infty) = -m b v_0 \vec{u}_z}$$

b) $\vec{\sigma}_O(S) = \vec{OS} \wedge m\vec{v}_S$

$\vec{OM} = r\vec{u}_r \qquad \vec{v} = r\dot{\theta}\vec{u}_\theta + \dot{r}\vec{u}_r \qquad$ ⚠ $\dot{\theta} < 0$.

Pour $\theta = \theta_S \quad \dot{r} = 0 \quad \vec{v} = -v_S \vec{u}_\theta$

$$\boxed{\vec{\sigma}_O(S) = -m r_{min} v_S \vec{u}_z}$$

3) Conservation de σ_O $\longrightarrow$ $\boxed{r_{min} v_S = b v_0}$

Conservation de E_m

$$\frac{1}{2}m v_0^2 - \underbrace{\frac{\mathcal{G}M_T m}{d}}_{\to 0} = \frac{1}{2}m v_S^2 - \frac{\mathcal{G}M_T m}{r_{min}}$$

$$\frac{1}{2}m v_0^2 = \frac{1}{2}m\frac{b^2 v_0^2}{r_{min}^2} - \frac{\mathcal{G}M_T m}{r_{min}}$$

$$r_{min}^2 + \frac{2\mathcal{G}M_T}{v_0^2}r_{min} - b^2 = 0.$$

$$\Delta = \frac{4\mathcal{G}^2 M_T^2}{v_0^4} + 4b^2$$

$$r_{min} = -\frac{\mathcal{G}M_T}{v_0^2} \cancel{\pm} + \sqrt{\frac{\mathcal{G}^2 M_T^2}{v_0^4} + b^2}$$

Il faut $r_{min} > R_T$

$$\frac{\mathcal{G}M_T}{v_0^2}\left[\sqrt{1 + \frac{b^2 v_0^4}{\mathcal{G}^2 M_T^2}} - 1\right] > R_T.$$

$$\Longleftrightarrow \quad 1 + \frac{b^2 v_0^4}{\mathcal{G}^2 M_T^2} > \left(\frac{R_T v_0^2}{\mathcal{G}M_T} + 1\right)^2$$

$$b_{min}^2 = \left[\left(\frac{R_T v_0^2}{\mathcal{G}M_T} + 1\right)^2 - 1\right] \times \frac{\mathcal{G}^2 M_T^2}{v_0^4}$$

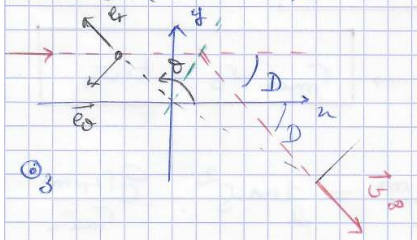
AN $\quad b_{min} = 9{,}1 \cdot 10^3$ km.

Astuce !

PFD $\quad \frac{d\vec{p}}{dt} = -\frac{\mathcal{G}M_T m}{r^2}\vec{e}_r \qquad$ or $\frac{d\vec{e}_\theta}{dt} = -\dot{\theta}\,\vec{e}_r$

soit $\quad \frac{d\vec{p}}{dt} = \frac{\mathcal{G}M_T m}{r^2\dot{\theta}}\frac{d\vec{e}_\theta}{dt} = -\frac{\mathcal{G}M_T m}{m b v_0}\frac{d\vec{e}_\theta}{dt}$

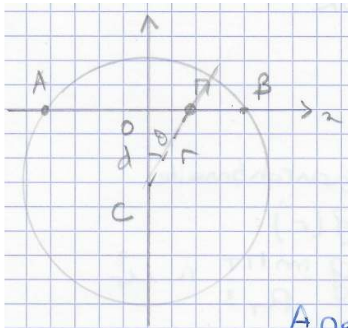
$$m(\vec{v}_\infty - \vec{v}_0) = -\frac{\mathcal{G}M_T}{b v_0}(\vec{e}_{\theta\infty} - \vec{e}_{\theta 0})$$



Selon $\vec{u}_x \quad v_0\cos D - v_0 = \frac{\mathcal{G}M_T}{m b v_0}\sin D$

Selon $\vec{u}_y \quad +v_0 \sin D = +\frac{\mathcal{G}M_T}{m b v_0}(\cos D - 1)$

11. Train dans un tunnel – MT



1. Th de Gauss

$$\Phi(\vec E) = \frac{q_{int}}{\varepsilon_0}$$

$\vec E$ = champ électrique

$$\vec F_{q_1 \to q_2} = \frac{q_1 q_2}{4\pi\varepsilon_0} \cdot \frac{1}{r^2}\vec u_{1\to 2}$$

Analogie $\vec g$ = champ gravitationnel

$$\vec F_{m_1 \to m_2} = -\mathcal{G}\frac{m_1 m_2}{r^2}\vec u_{1\to 2}$$

$q \leftrightarrow m$

$\frac{1}{4\pi\varepsilon_0} \leftrightarrow -\mathcal{G}$.

$$\Phi(\vec G) = -m_{int}\cdot \mathcal{G}\, 4\pi.$$

Pour $r > R_T$ $m_{int} = M_T$

$$\Phi(\vec G) = \iint \vec G(r)\cdot d\vec S = G(r)\cdot 4\pi r^2$$

Il vient $G(r) = -\frac{\mathcal{G} M_T}{r^2}$ ok.

Pour $r < R_T$ $m_{int} = \frac{M_T}{\frac{4}{3}\pi R_T^3} \times \frac{4}{3}\pi r^3$

$$= M_T \times \frac{r^3}{R_T^3}$$

$$G(r) \times \cancel{4\pi r^2} = -\cancel{4\pi}\,\mathcal{G} M_T \frac{r^{\cancel{3}}}{R_T^3}$$

$$G(r) = -\mathcal{G} M_T \frac{r}{R_T^3}$$

Pour $r = R_T$ $|G(R_T)| = \frac{\mathcal{G} M_T}{R_T^2}$

$|G(R_T)| = 9{,}8\ \mathrm{ms^{-2}}$ ok.

2. Σ = train.

Ref terrestre galiléen.

BdF → Interaction gravitationnelle

$$\vec F_g = m\vec g(r) = -\mathcal{G}\frac{m M_T}{R_T^3}\, r\,\vec u_r = -m g_0 \frac{r}{R_T}\vec u_r$$

→ $\vec N$ force de "guidage"

$\vec F_g$ force conservative.

$$E_m = \frac{1}{2}mv^2 + E_{pg}$$

$$\delta W(\vec F_g) = -m g_0 \frac{r}{R_T}\vec u_r \cdot dx\,\vec u_x$$

avec $r = \sqrt{d^2 + x^2}$ et $\vec u_r \cdot \vec u_x = \sin\theta = \frac{x}{r}$

$$= -m g_0\, x\,dx \cdot \frac{1}{R_T} = -dE_{pg}$$

$$E_{pg}(x) = \frac{1}{2} m g_0 \frac{x^2}{R_T}$$

$$\frac{1}{\cancel{2}}\cancel{m}\dot x^2 + \frac{1}{\cancel{2}}\cancel{m} g_0 \frac{x^2}{R_T} = \frac{1}{\cancel{2}}\cancel{m} g_0 R_T$$

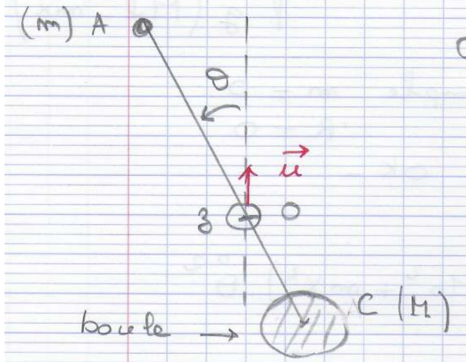
$$\left(\frac{dx}{dt}\right)^2 = R_T g_0 \left[1 - \frac{x^2}{R_T^2}\right]$$

$$\vec F_g \cdot \vec u_x = -m g_0 \frac{x}{R_T}$$

PFD $m\ddot x = -m g_0 \frac{x}{R_T}$

$$\ddot x + \frac{g_0}{R_T}x = 0 \qquad \frac{T}{2} = \frac{2\pi}{2\sqrt{g_0/R_T}} = \pi\sqrt{\frac{R_T}{g_0}}$$

12. Métronome – CCINP



$OC = \ell > X = OA$

$M > \ell$

Σ = métronome = {tige + boule + ponctuelle}

1) $L_\Delta(\Sigma) = (Mb^2 + mX^2)\,\dot{\theta}$

2) Bilan des actions

→ liaison pivot

→ $\vec{P}_A = -mg\,\vec{e_x}$

→ $\vec{P}_C = -Mg\,\vec{e_x}$

$$\vec{\mathcal{M}}_O(\vec{P}_A) = \vec{OA} \wedge m\vec{g} = X\vec{e_r} \wedge (-mg\,\vec{e_x}) = +Xmg\sin\theta\,\vec{e_z}$$

$$\vec{\mathcal{M}}_O(\vec{P}_C) = -\ell Mg\sin\theta\,\vec{e_z}$$

TMC $(Mb^2 + mX^2)\,\ddot{\theta} = g(Xm - \ell M)\sin\theta$

Petites oscillations $\ddot{\theta} + \omega_0^2\theta = 0$

$$\omega_0^2 = g\,\frac{\ell M - Xm}{Mb^2 + mX^2} > 0$$

13. Mesure de couple