Mardi 14 juin 2022

Ex 147.

a. Obtenir au moins un doublet requiert qu'il y en ait eu un premier :

$B_n = A_2 \cup \dots \cup A_n$ union disj $\quad P(B_n) = \sum_{k=2}^{n} p_k = \sum_{k=1}^{n} p_k$

$(p_1 = 0)$.

b.

$$\underbrace{1 \quad \cdots \quad n}_{\text{pas de doublet}} \quad \underset{n+1}{F} \quad \underset{n+2}{P} \quad \underset{n+3}{P}$$

$A_{n+3} = \overline{B_n} \cap F_{n+1} \cap P_{n+2} \cap P_{n+3}$ $\quad$ ind : $p_{n+3} = p^2 q \left(1 - \sum_{k=1}^{n} p_k\right)$

c. $p_{n+2} = p^2 q \left(1 - \sum_{k=1}^{n-1} p_k\right)$ donc $p_{n+3} - p_{n+2} = -p^2 q\, p_n$.

d. $U_n = \begin{bmatrix} p_n \\ p_{n+1} \\ p_{n+2} \end{bmatrix}$. $\quad U_{n+1} = \begin{bmatrix} p_{n+1} \\ p_{n+2} \\ p_{n+2} - p^2 q\, p_n \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -p^2 q & 0 & 1 \end{bmatrix}}_{\text{notée } A} U_n$.

$$\det(tI_3 - A) = \begin{vmatrix} t & -1 & 0 \\ 0 & t & -1 \\ p^2 q & 0 & t-1 \end{vmatrix} \underset{\text{dév } L_3}{=} p^2 q \begin{vmatrix} -1 & 0 \\ t & -1 \end{vmatrix} + (t-1)\begin{vmatrix} t & -1 \\ 0 & t \end{vmatrix} = t^3 - t^2 + p^2 q$$
$$= t^3 - t^2 + p^2 - p^3.$$

$\chi_A(t) = (t-p)\left((t^2 + tp + p^2) - (t+p)\right) = (t-p)\left(t^2 - (1-p)t - p(1-p)\right)$.

$\Delta = (1-p)^2 + 4p(1-p) = (1-p)(1+3p) > 0$. $\quad t_1 = \dfrac{1-p+\sqrt{\Delta}}{2}$

$\quad t_2 = \dfrac{1-p-\sqrt{\Delta}}{2}$.

$p_n = \alpha p^n + \beta t_1^n + \gamma t_2^n$.

Je ne vois guère quoi faire de plus.

Bof.

Ex 148. $X_i = \begin{cases} 1 & \text{si } i \text{ est pioché} \\ 0 & \text{sinon} \end{cases}$. Il y a 2^{n-1} poignées qui contiennent i donc $X_i \hookrightarrow \mathcal{B}(\frac{1}{2})$.

Si $i \neq j$
Il y a 2^{n-2} poignées qui contiennent i et j donc $\mathbb{P}(X_i = 1, X_j = 1) = 1/4$

X_i et X_j sont indep. $X = X_1 + 2X_2 + \dots + nX_n$.

$$\mathbb{E}(X) = \sum_{k=1}^{n} k\,\mathbb{E}(X_k) = \frac{n(n+1)}{4}.$$

$$\mathbb{V}(X) = \sum_{k=1}^{n} k^2\, \mathbb{V}(X_k) = \frac{n(2n+1)(n+1)}{24}.$$

Ex 149. $Z(\Omega) = [\![0,n]\!]$.

$[Z=0] = [X=0, Y=0]$ $\quad \mathbb{P}(Z=0) = (1-p)^n \times \frac{1}{n+1}$.

Soit $k \in [\![1,n]\!]$. $[Z=k] = [X=0, Y=k] \underset{\text{disj}}{\cup} [X=k]$

$$\mathbb{P}(Z=k) = (1-p)^n \times \frac{1}{n+1} + \binom{n}{k} p^k (1-p)^{n-k}.$$

$$\mathbb{E}(Z) = \sum_{k=0}^{n} k\,\mathbb{P}(Z=k) = \sum_{k=1}^{n} k \left[\quad \right]$$

$$= \frac{n}{2}(1-p)^n + \underbrace{\sum_{k=1}^{n} k\,\mathbb{P}(X=k)}_{=\mathbb{E}(X)} = \frac{n}{2}(1-p)^n + np.$$

Ex 151. Fait en classe : $Y \hookrightarrow \mathcal{P}(\lambda p)$. Probas totales :

$$\mathbb{P}[Y=k] = \sum_{n=0}^{+\infty} \mathbb{P}(X=n)\,\mathbb{P}(Y=k \mid X=n)$$

nulle si $n<k$
binomiale sinon

$$= \sum_{n=k}^{+\infty} \frac{e^{-\lambda}\lambda^n}{n!}\binom{n}{k} p^k (1-p)^{n-k} = \sum_{n=k}^{+\infty} \frac{e^{-\lambda}\lambda^n}{(n-k)!\,k!} p^k (1-p)^{n-k}$$

$$\underset{l=n-k}{=} \sum_{l=0}^{+\infty} \frac{e^{-\lambda}\lambda^{l+k}}{l!\,k!} p^k (1-p)^l = \frac{e^{-\lambda}\lambda^k}{k!} p^k e^{\lambda(1-p)} = \frac{e^{-\lambda p}(\lambda p)^k}{k!}$$

Ex 156. $M_n(\Omega) = [\![1, p]\!]$.

$[M_n \geq k] = \bigcap_{i=1}^{n} [U_i \geq k]$ donc $\mathbb{P}(M_n \geq k) = \left(\frac{p-k+1}{p}\right)^n$.

$$\mathbb{E}(M_n) = \sum_{k=1}^{+\infty} \mathbb{P}(M_n \geq k) = \sum_{k=1}^{p} \left(\frac{p-k+1}{p}\right)^n \underset{\ell = p+1-k}{=} \sum_{\ell=1}^{p} \left(\frac{\ell}{p}\right)^n.$$

$$= 1 + \sum_{\ell=1}^{p-1} \left(\frac{\ell}{p}\right)^n \xrightarrow[n \to +\infty]{} 1.$$

Ex 106 $M_n(\Omega) = [\![1, p]\!]$.

$[M_n \leq k] = \bigcap_{i=1}^{n} [U_i \leq k]$ donc $\mathbb{P}(M_n \leq k) = \left(\frac{k}{p}\right)^n$ (valable aussi si $k=0$).

et $\cancel{\mathbb{P}}$ $[M_n \leq k] = [M_n \leq k-1] \underset{\text{disj}}{\cup} [M_n = k]$

$\forall k \in [\![1, p]\!]$, $\mathbb{P}(M_n = k) = \left(\frac{k}{p}\right)^n - \left(\frac{k-1}{p}\right)^n$.

$$\mathbb{E}(M_n) = \sum_{k=1}^{+\infty} \underbrace{\mathbb{P}(M_n \geq k)}_{=0 \text{ si } k>p} = \sum_{k=1}^{p} \left(1 - \mathbb{P}(M_n < k)\right).$$

$\ell = k-1 \longrightarrow$

$$= p - \sum_{\ell=1}^{p-1} \left(\frac{\ell}{p}\right)^n \xrightarrow[n \to +\infty]{} p.$$

Ex 157: 1. $\mathbb{P}(X_i > n) = (1-p)^n$. 2. $(Y_N > n) = \bigcap_{i=1}^{N} (X_i > n)$ et $\mathbb{P}(Y_N > n) = (1-p)^{nN}$.

3. $$\mathbb{E}(Y_N) = \sum_{k=1}^{+\infty} \mathbb{P}(Y_N \geq k) = \sum_{\ell=0}^{+\infty} \mathbb{P}(Y_N > \ell) = \sum_{\ell=0}^{+\infty} (1-p)^{\ell N} = \frac{1}{1-(1-p)^N}.$$