Jeudi 20 juin 2024

Ex 97. $f_n(x) = \begin{cases} \left(1+\frac{x}{n}\right)^n e^{-2x} & \text{si } x \in [0,n] \\ 0 & \text{si } x \geq n \end{cases}$

$I_n = \int_0^{+\infty} f_n$.

1 Soit $x \geq 0$. Pour tout $\overset{\text{entier}}{n} \geq x$, $f_n(x) = \exp\Big[\underbrace{n\ln\left(1+\frac{x}{n}\right) - 2x}_{\text{ceci tend vers } -x}\Big]$

donc $\boxed{f_n(x) \xrightarrow[n\to+\infty]{} e^{-x}.}$

2 si $n \geq x$, $0 \leq f_n(x) \leq \exp\left(n \times \frac{x}{n} - 2x\right) = e^{-x}$

si $n < x$, $0 \leq f_n(x) = 0 \leq e^{-x}$ aussi.

$\forall n \in \mathbb{N}^*$, $\forall x \geq 0$, $|f_n(x)| \leq e^{-x}$ int.

Th conv dom : $I_n \xrightarrow[n\to+\infty]{} \int_0^{+\infty} e^{-x}dx = 1$.

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Ex 116. ① ~~Dans~~ On pose $n = \text{Card}(A)$.

Dans $\mathbb{R}_{n-1}[X]$, l'unique sol est $Q_0 = \prod_{x\in A\setminus\{x_0\}} \frac{X-x}{x_0-x}$.

~~L'ensemble~~ On pose $P = \prod_{x\in A}(X-x)$. L'ens des sol est $\{Q_0 + PR ; R \in \mathbb{R}[X]\}$.

② $\overset{\text{Connu}}{\Longrightarrow}$. On suppose que pour tout $(k,l) \in \mathbb{N}^2$, $\mathbb{E}(X^kY^l) = \mathbb{E}(X^k)\mathbb{E}(Y^l)$.

On en déduit : $\forall (P,Q) \in \mathbb{R}[X]^2$, $\mathbb{E}(P(X)Q(Y)) = \mathbb{E}(P(X))\mathbb{E}(Q(Y))$.

Etant donné $(x_0, y_0) \in X(\Omega) \times Y(\Omega)$, on choisit $P_0$ et $R_0$

comme en Q1, de sorte que ~~$P(X)(\Omega) = \{0,1\}$~~

$Q_0(X) = \begin{cases} 1 & \text{si } X = x_0 \\ 0 & \text{sinon} \end{cases}$, $R_0(Y) = \begin{cases} 1 & \text{si } Y = y_0 \\ 0 & \text{sinon} \end{cases}$.

Alors $(P(X)Q(Y) = 1) = (X = x_0, Y = y_0)$ cette formule donne

$\mathbb{P}(X = x_0, Y = y_0) = \mathbb{P}(X = x_0)\mathbb{P}(Y = y_0)$.

# Ex 105.

① $\varphi_n$ est cont, str cr, avec $\begin{cases} \varphi_n(0)=0 \\ \varphi_n(1)=1. \end{cases}$

② $M_p = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{pmatrix}$ $\quad rg(M_p) = 1.$

$\Updownarrow$

$\mathrm{Im}(M_p) = \mathrm{Vect}\left(\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}\right).$

$M_p \in S_p(\mathbb{R})$ donc $M_p$ est diag.

③ $\dim(\mathrm{Ker}(M_p)) = p-1.$ $\quad \mathrm{mult}(0, M_p) \geq \dim(E_0(M_p)) = p-1$

donc $X^{p-1} \mid \chi_{M_p}$. $\quad$ Or $\chi_{M_p} = X^p - tr(M_p)X^{p-1} + \text{etc}$

donc etc $= 0$ $\quad$ et $\chi_{M_p} = X^p - tr(M_p)X^{p-1} = X^{p-1}(X-p).$

$E_p(M_p) = \mathrm{Vect}\left(\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}\right).$ $\quad P^{-1} M_p P = \begin{bmatrix} 0 & & 0 \\ & \ddots & \\ 0 & & n \end{bmatrix}$

$P^{-1}(M_p + I_p)P = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ & & 1 \\ 0 & & n+1 \end{bmatrix}$ $\quad Sp(M_p + I_p) = \{1, n+1\}.$

④ $f_p$ est polynomiale. ~~si $i \in [\![1, p-1]\!]$~~ $\quad \dfrac{\partial f_p}{\partial x_i}(x) = n x_i^{n-1} - n\left(1 - \sum_{k=1}^{p} x_k\right)^{n-1}$

⑤ $Df_p(x) = 0 \iff \begin{cases} x_1 = 1 - \sum_{k=1}^{p} x_k \\ \vdots \\ x_p = 1 - \sum_{k=1}^{p} x_k \end{cases} \underset{L_p \leftarrow \sum_{i=1}^{p} L_i}{\iff} \begin{cases} x_1 = 1 - \Sigma \\ \vdots \\ x_{p-1} = 1 - \Sigma \\ \sum_{k=1}^{p} x_k = p - p\sum_{k=1}^{p} x_k \end{cases}$

$\iff \begin{cases} x_1 = 1 - \Sigma \\ \vdots \\ x_{p-1} = 1 - \Sigma \\ \sum_{k=1}^{p} x_k = \frac{p}{p+1} \end{cases} \iff \begin{cases} x_1 = \frac{1}{p+1} \\ \vdots \\ x_{p-1} = \frac{1}{p+1} \\ x_p = \frac{1}{p+1} \end{cases}$ $\quad M_p = \frac{1}{p+1}(1, \ldots, 1)$

⑥ $\dfrac{\partial^2 f_p}{\partial x_i^2}(x) = n(n-1)x_i^{n-2} + n(n-1)\left(1 - \sum_{k=1}^{p} x_k\right)^{n-2}$ $\quad \dfrac{\partial^2 f_p}{\partial x_i \partial x_j}(x) = n(n-1)\left(1 - \sum_{k=1}^{p} x_k\right)^{n-2}.$

$\dfrac{\partial^2 f_p}{\partial x_i^2}(M_p) = n(n-1)\dfrac{1}{(p+1)^{n-2}} \times 2$ $\quad \dfrac{\partial^2 f_p}{\partial x_i \partial x_j}(M_p) = \dfrac{n(n-1)}{(p+1)^{n-2}}.$

$H_f(M_p) = n(n+1)\dfrac{1}{(p+1)^{n-2}}(I_p + M_p) \in S_p^{++}(\mathbb{R})$ donc $f_p$ a un min local en $a_p$.