Formulaire d'analyse vectorielle

Soit $f(x,y,z,t)$ un champ scalaire, $\overrightarrow{A}(x,y,z,t) = A_x(x,y,z,t)\,\vec{u_x} + A_y(x,y,z,t)\,\vec{u}_y + A_z(x,y,z,t)\,\vec{u}_z$ et $\overrightarrow{B}(x,y,z,t) = B_x(x,y,z,t)\,\vec{u_x} + B_y(x,y,z,t)\,\vec{u}_y + B_z(x,y,z,t)\,\vec{u}_z$ deux champs vectoriels.

I - Les opérateurs dans les différents systèmes de coordonnées

1) Coordonnées cartésiennes

• Opérateur nabla :
$$\overrightarrow{\nabla} = \frac{\partial}{\partial x}\vec{u}_x + \frac{\partial}{\partial y}\vec{u}_y + \frac{\partial}{\partial z}\vec{u}_z.$$

• Opérateur gradient :
scalaire → vecteur
On a $\overrightarrow{\text{grad}}\,f \cdot \overrightarrow{\text{d}\ell} = \text{d}f$
$$\overrightarrow{\text{grad}}\,f = \overrightarrow{\nabla} f = \frac{\partial f}{\partial x}\vec{u}_x + \frac{\partial f}{\partial y}\vec{u}_y + \frac{\partial f}{\partial z}\vec{u}_z.$$

• Opérateur divergence :
vecteur → scalaire
$$\text{div}\,\overrightarrow{A} = \overrightarrow{\nabla}\cdot\overrightarrow{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}.$$

• Opérateur rotationnel :
vecteur → vecteur
$$\overrightarrow{\text{rot}}\,\overrightarrow{A} = \overrightarrow{\nabla}\wedge\overrightarrow{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\vec{u}_x + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\vec{u}_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\vec{u}_z$$

• Opérateur laplacien scalaire:
scalaire → scalaire
$$\Delta f = \overrightarrow{\nabla}\cdot\overrightarrow{\nabla} f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

• Opérateur laplacien vectoriel :
vecteur → vecteur
$$\Delta\overrightarrow{A} = \Delta A_x\,\vec{u}_x + \Delta A_y\,\vec{u}_y + \Delta A_z\,\vec{u}_z.$$

• "A scalaire gradient" appliqué à un scalaire : $\left(\overrightarrow{A}\cdot\overrightarrow{\text{grad}}\right) f = \overrightarrow{A}\cdot\left(\overrightarrow{\text{grad}}\,f\right) = A_x\frac{\partial f}{\partial x} + A_y\frac{\partial f}{\partial y} + A_z\frac{\partial f}{\partial z}$

Appliqué à un vecteur : $\left(\overrightarrow{A}\cdot\overrightarrow{\text{grad}}\right)\overrightarrow{B} = \left(A_x\frac{\partial}{\partial x} + A_y\frac{\partial}{\partial y} + A_z\frac{\partial}{\partial z}\right)\overrightarrow{B} = \left(A_x\frac{\partial B_x}{\partial x} + A_y\frac{\partial B_x}{\partial y} + A_z\frac{\partial B_x}{\partial z}\right)\vec{u}_x + \left(A_x\frac{\partial B_y}{\partial x} + A_y\frac{\partial B_y}{\partial y} + A_z\frac{\partial B_y}{\partial z}\right)\vec{u}_y + \left(A_x\frac{\partial B_z}{\partial x} + A_y\frac{\partial f}{\partial y} + A_z\frac{\partial B_z}{\partial z}\right)\vec{u}_z$

2) Coordonnées cylindriques

$$\overrightarrow{\text{grad}}\,f = \frac{\partial f}{\partial r}\vec{u}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\vec{u}_\theta + \frac{\partial f}{\partial z}\vec{u}_z.$$

$\text{div}\,\overrightarrow{A} = \frac{1}{r}\frac{\partial(rA_r)}{\partial r} + \frac{1}{r}\frac{\partial A_\theta}{\partial\theta} + \frac{\partial A_z}{\partial z}$. (pas à savoir)

$\overrightarrow{\text{rot}}\,\overrightarrow{A} = \left(\frac{1}{r}\frac{\partial A_z}{\partial\theta} - \frac{\partial A_\theta}{\partial z}\right)\vec{u}_r + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\vec{u}_\theta + \frac{1}{r}\left(\frac{\partial(rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial\theta}\right)\vec{u}_z$. (pas à savoir)

$\Delta f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2} + \frac{\partial^2 f}{\partial z^2}$. (pas à savoir)

3) Coordonnées sphériques

$\overrightarrow{\text{grad}} f = \frac{\partial f}{\partial r}\,\vec{u}_r + \frac{1}{r}\,\frac{\partial f}{\partial \theta}\,\vec{u}_\theta + \frac{1}{r\sin\theta}\,\frac{\partial f}{\partial \varphi}\,\vec{u}_\varphi.$

$\text{div}\,\overrightarrow{A} = \frac{1}{r^2}\,\frac{\partial (r^2 A_r)}{\partial r} + \frac{1}{r\sin\theta}\,\frac{\partial (\sin\theta A_\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\,\frac{\partial A_\varphi}{\partial \varphi}.$ (pas à savoir)

$\overrightarrow{\text{rot}}\,\overrightarrow{A} = \frac{1}{r\sin\theta}\left(\frac{\partial (\sin\theta A_\varphi)}{\partial\theta} - \frac{\partial A_\theta}{\partial\varphi}\right)\vec{u}_r + \left(\frac{1}{r\sin\theta}\,\frac{\partial A_r}{\partial\varphi} - \frac{1}{r}\,\frac{\partial (r A_\varphi)}{\partial r}\right)\vec{u}_\theta + \frac{1}{r}\left(\frac{\partial (r A_\theta)}{\partial r} - \frac{\partial A_r}{\partial\theta}\right)\vec{u}_\varphi.$ (pas à savoir)

$\Delta f = \frac{1}{r^2}\,\frac{\partial}{\partial r}\left(r^2\,\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\,\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin\theta}\,\frac{\partial^2 f}{\partial\varphi^2}.$ (pas à savoir)

II - Composition d'opérateurs

$$\overrightarrow{\text{rot}}(\overrightarrow{\text{grad}}\,f) = \vec{0} \qquad \left(\overrightarrow{\nabla} \wedge (\overrightarrow{\nabla} f) = \vec{0}\right)$$

$$\text{div}(\overrightarrow{\text{rot}}\,\overrightarrow{A}) = 0 \qquad \left(\overrightarrow{\nabla} \cdot (\overrightarrow{\nabla} \wedge \overrightarrow{A}) = \vec{0}\right)$$

$$\text{div}(\overrightarrow{\text{grad}}\,f) = \Delta f \qquad \left(\overrightarrow{\nabla} \cdot (\overrightarrow{\nabla} f) = \Delta f\right)$$

$$\overrightarrow{\text{rot}}(\overrightarrow{\text{rot}}\,\overrightarrow{A}) = \overrightarrow{\text{grad}}(\text{div}\,\overrightarrow{A}) - \Delta\overrightarrow{A} \qquad \left(\overrightarrow{\nabla} \wedge (\overrightarrow{\nabla} \wedge \overrightarrow{A}) = \overrightarrow{\nabla}(\overrightarrow{\nabla} \cdot \overrightarrow{A}) - \Delta\overrightarrow{A}\right)$$

III - Opérateurs et produits

$$\overrightarrow{\text{grad}}(fg) = f\,\overrightarrow{\text{grad}}\,g + g\,\overrightarrow{\text{grad}}\,f$$

$$\text{div}(f\overrightarrow{A}) = f\,\text{div}\,\overrightarrow{A} + \overrightarrow{\text{grad}}\,f \cdot \overrightarrow{A}$$

$$\overrightarrow{\text{rot}}(f\overrightarrow{A}) = f\,\overrightarrow{\text{rot}}\,\overrightarrow{A} + \overrightarrow{\text{grad}}\,f \wedge \overrightarrow{A}$$

$$\text{div}(\overrightarrow{A} \wedge \overrightarrow{B}) = \overrightarrow{B} \cdot \overrightarrow{\text{rot}}\,\overrightarrow{A} - \overrightarrow{A} \cdot \overrightarrow{\text{rot}}\,\overrightarrow{B}$$

$$\left(\overrightarrow{A} \cdot \overrightarrow{\text{grad}}\right)\overrightarrow{A} = \frac{1}{2}\,\overrightarrow{\text{grad}}\|\overrightarrow{A}\|^2 + \left(\overrightarrow{\text{rot}}\overrightarrow{A}\right) \wedge \overrightarrow{A}$$