ex. 10

On cherche $z^2 = 1 + i$, avec $z \in \mathbb{C}$.

① (par la forme algébrique).

Soit $z \in \mathbb{C}$, $\exists (\alpha, \beta) \in \mathbb{R}^2$, $z = \alpha + i\beta$.

On a,

$$z^2 = 1 + i \iff (\alpha + i\beta)^2 = 1 + i$$
$$\iff \alpha^2 - \beta^2 + i2\alpha\beta = 1 + i$$
$$\iff \begin{cases} \alpha^2 - \beta^2 = 1 \\ 2\alpha\beta = 1 \end{cases} \text{ par identification}$$

De plus,

$$|z^2| = \alpha^2 + \beta^2$$
$$\text{et } |1+i| = \sqrt{2}$$

D'où $z^2 = 1 + i \iff \begin{cases} \alpha^2 - \beta^2 = 1 & (1) \\ 2\alpha\beta = 1 & (2) \\ \alpha^2 + \beta^2 = \sqrt{2} & (3) \end{cases}$

$$\iff \begin{cases} 2\alpha^2 = 1 + \sqrt{2} & (1)+(3) \\ \alpha\beta = \frac{1}{2} \\ \alpha^2 + \beta^2 = \sqrt{2} \end{cases}$$

$$\iff \begin{cases} \alpha^2 = \frac{1+\sqrt{2}}{2} \\ \alpha\beta = \frac{1}{2} \\ \beta^2 = \sqrt{2} - \frac{1+\sqrt{2}}{2} = \frac{\sqrt{2}-1}{2} \end{cases}$$

$\hat{c}$ $\alpha\beta > 0$, $\iff \begin{cases} \alpha = \sqrt{\frac{1+\sqrt{2}}{2}} \\ \text{et} \\ \beta = \sqrt{\frac{\sqrt{2}-1}{2}} \end{cases}$ ou $\begin{cases} \alpha = -\sqrt{\frac{1+\sqrt{2}}{2}} \\ \text{et} \\ \beta = -\sqrt{\frac{\sqrt{2}-1}{2}} \end{cases}$

Donc, $\boxed{z_1 = \sqrt{\frac{1+\sqrt{2}}{2}} + i\sqrt{\frac{\sqrt{2}-1}{2}}}$

et $\boxed{z_2 = -\sqrt{\frac{1+\sqrt{2}}{2}} - i\sqrt{\frac{\sqrt{2}-1}{2}}}$

② (par la forme exp.)

On a, $1+i = \sqrt{2}\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)$
$= \sqrt{2}\, e^{i\pi/4}$

Soit $(z,\rho,\theta) \in \mathbb{C} \times (\mathbb{R}_+^*) \times \mathbb{R}$, tq $z = \rho e^{i\theta}$

On a,

$$z^2 = 1+i \iff (\rho e^{i\theta})^2 = \sqrt{2}\, e^{i\pi/4}$$
$$\iff \rho^2 e^{i2\theta} = \sqrt{2}\, e^{i\pi/4}$$

$$\iff \begin{cases} \rho^2 = \sqrt{2} \\ 2\theta \equiv \frac{\pi}{4}\ [2\pi] \end{cases}$$ par identification

On a $\rho > 0$, donc $\rho = \sqrt[4]{2}$.

De plus, $e^{i\pi} = -1$, donc

$$z^2 = 1+i \iff \boxed{\begin{cases} z_1 = \sqrt[4]{2}\, e^{i\frac{\pi}{8}} \\ \text{et} \\ z_2 = -\sqrt[4]{2}\, e^{i\frac{\pi}{8}} \end{cases}}$$

On a donc,

$$\sqrt{\frac{\sqrt{2}+1}{2}} + i\sqrt{\frac{\sqrt{2}-1}{2}} = \sqrt[4]{2}\cos\frac{\pi}{8} + i\sqrt[4]{2}\sin\frac{\pi}{8}$$

$$\iff \begin{cases} \sqrt[4]{2}\cos\frac{\pi}{8} = \sqrt{\frac{\sqrt{2}+1}{2}} \\ \sqrt[4]{2}\sin\frac{\pi}{8} = \sqrt{\frac{\sqrt{2}-1}{2}} \end{cases}$$

$$\iff \boxed{\begin{cases} \cos\frac{\pi}{8} = \sqrt{\frac{\sqrt{2}+1}{2}} \times \frac{1}{\sqrt[4]{2}} = \frac{\sqrt[4]{2}\sqrt{1+\sqrt{2}}}{2} \\ \sin\frac{\pi}{8} = \frac{\sqrt[4]{2}\sqrt{\sqrt{2}-1}}{2} \end{cases}}$$