Corrigé de la liste d'exercices n°3 **Sommes et produits**

Exercice 1

1. $\displaystyle\sum_{i=1}^{n} 1 = n.$
2. $\displaystyle\sum_{i=1}^{n}\sum_{j=1}^{n} 1 = \sum_{i=1}^{n} n = n^2.$
3. $\displaystyle\sum_{i=1}^{n}\sum_{j=1}^{i} 1 = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$
4. $\displaystyle\sum_{i=1}^{n}\sum_{j=1}^{n}(i+j) = \sum_{i=1}^{n}\left(\sum_{j=1}^{n} i + \sum_{j=1}^{n} j\right) = \sum_{i=1}^{n}\left(n\times i + \frac{n(n+1)}{2}\right) = \frac{n^2(n+1)}{2} + \frac{n^2(n+1)}{2} = n^2(n+1).$
5. $\displaystyle\sum_{i=1}^{n}\sum_{j=i}^{n} j = \sum_{j=1}^{n}\sum_{i=1}^{j} j = \sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}.$
6. $\displaystyle\sum_{k=0}^{n}\sum_{j=1}^{m} kj^2 = \left(\sum_{k=0}^{n} k\right)\left(\sum_{j=1}^{m} j^2\right) = \frac{n(n+1)m(m+1)(2m+1)}{12}.$
7.
$$\begin{aligned}
\sum_{k=0}^{n^2}\sum_{j=k}^{k+2} kj^2 &= \sum_{k=0}^{n^2} k(k^2+(k+1)^2+(k+2)^2)\\
&= \sum_{k=0}^{n^2} 3k^3+6k^2+5k\\
&= 3\frac{n^4(n^2+1)^2}{4} + n^2(n^2+1)(2n^2+1) + \frac{5n^2(n^2+1)}{2}\\
&= \frac{n^2(n^2+1)}{4}(3n^2(n^2+1)+8n^2+4+10)\\
&= \frac{n^2(n^2+1)(3n^4+11n^2+14)}{4}
\end{aligned}$$

Exercice 2

1. $\displaystyle\sum_{k=0}^{n} x^{2k+1} = x\sum_{k=0}^{n}(x^2)^k = \begin{cases} n+1 & \text{si } x=1\\ -n-1 & \text{si } x=-1\\ x\times\dfrac{1-x^{2n+2}}{1-x^2} & \text{si } x\notin\{-1,1\}.\end{cases}$
2. $\displaystyle\sum_{k=0}^{n} a^k 2^{3k} x^{-k} = \sum_{k=0}^{n}(8ax^{-1})^k = \begin{cases} n+1 & \text{si } x=8a\\ \dfrac{1-(8ax^{-1})^{n+1}}{1-8ax^{-1}} & \text{si } x\neq 8a.\end{cases}$
3. Si $x=1$, on a $\displaystyle\sum_{k=0}^{n}\sum_{j=0}^{k} x^j = \sum_{k=0}^{n}(k+1) = \sum_{k=1}^{n+1} k = \frac{(n+1)(n+2)}{2}.$

Supposons dorénavant que $x \neq 1$.

$$\begin{aligned}\sum_{k=0}^{n}\sum_{j=0}^{k} x^j &= \sum_{k=0}^{n}\frac{1-x^{k+1}}{1-x}\\ &= \frac{1}{1-x}\left(n+1-\sum_{k=0}^{n}x^{k+1}\right)\\ &= \frac{1}{1-x}\left(n+1-x\sum_{k=0}^{n}x^{k}\right)\\ &= \frac{1}{1-x}\left(n+1-x\times\frac{1-x^{n+1}}{1-x}\right).\end{aligned}$$

4. $\displaystyle\sum_{k=1}^{n}(-1)^k 2^{2k+1} = 2\sum_{k=1}^{n}(-4)^k = 2\times(-4)\times\frac{1-(-4)^n}{1-(-4)} = \frac{8}{5}((-4)^n-1).$
5. $\displaystyle\sum_{k=1}^{n}2^{2k+1}3^{2k-1} = \frac{2}{3}\sum_{k=1}^{n}36^k = \frac{2}{3}\times 36\times\frac{1-36^n}{1-36} = \frac{24}{35}(36^n-1).$
6. $\displaystyle\sum_{k=2}^{n+1}\frac{1}{(1-k)^3}+\frac{1}{k^3} = \sum_{k=2}^{n+1}\frac{1}{k^3}-\frac{1}{(k-1)^3} = \frac{1}{(n+1)^3}-1.$
7. $\displaystyle\sum_{k=1}^{n}\frac{k}{(k+1)!} = \sum_{k=1}^{n}\frac{k+1-1}{(k+1)!} = \sum_{k=1}^{n}\frac{1}{k!}-\frac{1}{(k+1)!} = 1-\frac{1}{(n+1)!}.$

Exercice 3

1. $\displaystyle S_n = \sum_{k=0}^{n}(-2a)^k = \begin{cases} n+1 & \text{si } a = -\frac{1}{2}.\\ \dfrac{1-(-2a)^{n+1}}{1+2a} & \text{si } a \neq -\frac{1}{2}.\end{cases}$
2. $\displaystyle T_n = \sum_{k=0}^{n-1}(e^{\frac{a}{n}})^k = \begin{cases} n & \text{si } a = 0.\\ \dfrac{1-e^a}{1-e^{\frac{a}{n}}} & \text{si } a \neq 0.\end{cases}$

Exercice 4

On a pour tout $n \in \mathbb{N}$;

$$\sum_{k=0}^{n}(2k+1) = 2\sum_{k=0}^{n}k+\sum_{k=0}^{n}1 = n(n+1)+n+1 = (n+1)^2.$$

Exercice 5

On pose pour tout $n \in \mathbb{N}, u_n = \dfrac{1}{2^n}$.

1.

$$\sum_{k=0}^{2n+1}u_k = \sum_{k=0}^{2n+1}\frac{1}{2^k} = \sum_{k=0}^{2n+1}\left(\frac{1}{2}\right)^k = \frac{1-\left(\dfrac{1}{2}\right)^{2n+2}}{1-\dfrac{1}{2}} = 2\left(1-\frac{1}{2^{2n+2}}\right).$$

2.

$$\sum_{k=1}^{2n} u_{2k} = \sum_{k=1}^{2n} \frac{1}{2^{2k}} = \sum_{k=1}^{2n} \frac{1}{4^k} = \frac{1}{4}\frac{1-\frac{1}{4^{2n}}}{1-\frac{1}{4}} = \frac{1}{3}\left(1-\frac{1}{4^{2n}}\right).$$

3. En posant le changement d'indice $i = 2n - k$, on a

$$\sum_{k=0}^{n} u_{2n-k} = \sum_{i=n}^{2n} u_i = \sum_{i=n}^{2n} \frac{1}{2^i} = \frac{1}{2^n}\frac{1-\frac{1}{2^{n+1}}}{1-\frac{1}{2}} = \frac{1}{2^{n-1}}\left(1-\frac{1}{2^{n+1}}\right).$$

4. En posant le changement d'indice $i = k + n$, on obtient

$$\sum_{k=0}^{n} u_{k+n} = \sum_{i=n}^{2n} u_i = \frac{1}{2^{n-1}}\left(1-\frac{1}{2^{n+1}}\right).$$

5.

$$\sum_{k=0}^{n} u_k + n = \sum_{k=0}^{n} \frac{1}{2^k} + n = \frac{1-\frac{1}{2^{n+1}}}{1-\frac{1}{2}} + n = 2\left(1-\frac{1}{2^{n+1}}\right) + n.$$

Exercice 6

1. (a) Par linéarité de la somme, on a

$$\begin{aligned}
\sum_{k=1}^{n}(k-1)^2 &= \sum_{k=1}^{n}(k^2-2k+1) \\
&= \sum_{k=1}^{n} k^2 - 2\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1 \\
&= \frac{n(n+1)(2n+1)}{6} - n(n+1) + n \\
&= \frac{n}{6}(2n^2+3n+1-6n) \\
&= \frac{n}{6}(2n^2-3n+1) \\
&= \frac{(n-1)n(2n-1)}{6}.
\end{aligned}$$

(b) On pose $i = k - 1$ et on obtient

$$\sum_{k=1}^{n}(k-1)^2 = \sum_{i=0}^{n-1} i^2 = \frac{(n-1)n(2(n-1)+1)}{6} = \frac{(n-1)n(2n-1)}{6}.$$

2. (a) Pour tout $k \in [\![1, n]\!]$, on a

$$(k+1)^4 - k^4 = ((k+1)^2-k^2)((k+1)^2+k^2) = (2k+1)(2k^2+2k+1) = 4k^3+6k^2+4k+1.$$

(b) D'après la question précédente, on a

$$\begin{aligned}
\sum_{k=0}^{n}(k+1)^4-k^4 &= \sum_{k=0}^{n}(4k^3+6k^2+4k+1)\\
\Leftrightarrow (n+1)^4 &= 4\sum_{k=0}^{n}k^3+6\sum_{k=0}^{n}k^2+4\sum_{k=0}^{n}k+\sum_{k=0}^{n}1\\
\Leftrightarrow (n+1)^4 &= 4\sum_{k=0}^{n}k^3+n(n+1)(2n+1)+2n(n+1)+(n+1)\\
\Leftrightarrow 4\sum_{k=0}^{n}k^3 &= (n+1)[(n+1)^3-2n^2-n-2n-1]\\
\Leftrightarrow 4\sum_{k=0}^{n}k^3 &= (n+1)(n^3+n^2)\\
\Leftrightarrow 4\sum_{k=0}^{n}k^3 &= n^2(n+1)^2
\end{aligned}$$

donc $\displaystyle\sum_{k=0}^{n}k^3=\frac{n^2(n+1)^2}{4}=\left(\frac{n(n+1)}{2}\right)^2=\left(\sum_{k=0}^{n}k\right)^2.$

Exercice 7

1. En utilisant la séparation des variables, on a

$$\begin{aligned}
\sum_{\substack{0\le i\le n\\0\le j\le p}} i(j^2+1) &= \left(\sum_{i=0}^{n}i\right)\left(\sum_{j=0}^{p}(j^2+1)\right)\\
&= \frac{n(n+1)}{2}\left(\sum_{j=0}^{p}j^2+\sum_{j=0}^{p}1\right)\\
&= \frac{n(n+1)}{2}\left(\frac{p(p+1)(2p+1)}{6}+p+1\right)\\
&= \frac{n(n+1)}{12}(p+1)(p(2p+1)+6)\\
&= \frac{n(n+1)}{12}(p+1)(2p^2+p+6).
\end{aligned}$$

2. En utilisant la séparation des variables, on a

$$\begin{aligned}
\sum_{0\le i,j\le n} 2^{i+j} &= \sum_{0\le i,j\le n} 2^i2^j\\
&= \left(\sum_{i=0}^{n}2^i\right)\left(\sum_{j=0}^{n}2^j\right)\\
&= \left(\sum_{i=0}^{n}2^i\right)^2\\
&= \left(\frac{1-2^{n+1}}{1-2}\right)^2\\
&= (2^{n+1}-1)^2.
\end{aligned}$$

3.

$$\begin{aligned}
\sum_{0\le i\le j\le n} 2^{i+j} &= \sum_{j=0}^{n} 2^j \sum_{i=0}^{j} 2^i \\
&= \sum_{j=0}^{n} 2^j \frac{1-2^{j+1}}{1-2} \\
&= \sum_{j=0}^{n} 2^j (2^{j+1}-1) \\
&= \sum_{j=0}^{n} 2^{2j+1} - 2^j \\
&= 2\sum_{j=0}^{n} 4^j - \sum_{j=0}^{n} 2^j \\
&= 2\frac{1-4^{n+1}}{1-4} - \frac{1-2^{n+1}}{1-2} \\
&= \frac{2}{3}(4^{n+1}-1) - 2^{n+1} + 1 \\
&= \frac{2}{3}(2^{n+1}-1)(2^{n+1}+1) - 2^{n+1} + 1 \\
&= (2^{n+1}-1)(\frac{2}{3}2^{n+1} - \frac{1}{3}) \\
&= \frac{1}{3}(2^{n+1}-1)(2^{n+2}-1).
\end{aligned}$$

4.

$$\begin{aligned}
\sum_{0\le i\le j\le n} \frac{i}{j+1} &= \sum_{j=0}^{n}\sum_{i=0}^{j} \frac{i}{j+1} \\
&= \sum_{j=0}^{n} \frac{1}{j+1} \sum_{i=0}^{j} i \\
&= \sum_{j=0}^{n} \frac{1}{j+1} \frac{j(j+1)}{2} \\
&= \frac{1}{2}\sum_{j=0}^{n} j \\
&= \frac{n(n+1)}{4}.
\end{aligned}$$

5.

$$\begin{aligned}
\sum_{i=1}^{n}\sum_{j=i}^{n} \frac{1}{j} &= \sum_{j=1}^{n}\sum_{i=1}^{j} \frac{1}{j} \\
&= \sum_{j=1}^{n} j \times \frac{1}{j} \\
&= \sum_{j=1}^{n} 1 \\
&= n.
\end{aligned}$$

Exercice 8

Soit $n \in \mathbb{N}^*$. On a

$$\begin{aligned}
\sum_{1 \leq i \leq j \leq n} ij &= \sum_{j=1}^{n} \sum_{i=1}^{j} ij \\
&= \sum_{j=1}^{n} j \sum_{i=1}^{j} i \\
&= \sum_{j=1}^{n} \frac{j^2(j+1)}{2} \\
&= \frac{1}{2} \sum_{j=1}^{n} j^3 + j^2 \\
&= \frac{1}{2} \left(\sum_{j=1}^{n} j^3 + \sum_{j=1}^{n} j^2 \right) \\
&= \frac{1}{2} \left(\frac{n^2(n+1)^2}{4} + \frac{n(n+1)(2n+1)}{6} \right) \\
&= \frac{1}{2} \left(\frac{3n^2(n+1)^2 + 2n(n+1)(2n+1)}{12} \right) \\
&= \frac{n(n+1)}{24} (3n(n+1) + 2(2n+1)) \\
&= \frac{n(n+1)}{24} (3n^2 + 7n + 2) \\
&= \frac{n(n+1)(n+2)(3n+1)}{24}.
\end{aligned}$$

Exercice 9

1. On a $\max(k,j) = \begin{cases} k & \text{si } k \geq j \\ j & \text{si } j \geq k \end{cases}$ d'où

$$\begin{aligned}
\sum_{k=1}^{n} \sum_{j=1}^{n} \max(k,j) &= \sum_{k=1}^{n} \left(\sum_{j=1}^{k-1} k + \sum_{j=k}^{n} j \right) \\
&= \sum_{k=1}^{n} \sum_{j=1}^{k-1} k + \sum_{k=1}^{n} \sum_{j=k}^{n} j \\
&= \sum_{k=1}^{n} k(k-1) + \sum_{j=1}^{n} \sum_{k=1}^{j} j \\
&= \sum_{k=1}^{n} k^2 - \sum_{k=1}^{n} k + \sum_{j=1}^{n} j^2 \\
&= \frac{n(n+1)(2n+1)}{3} - \frac{n(n+1)}{2} \\
&= \frac{n(n+1)(2(2n+1)-3)}{6} \\
&= \frac{n(n+1)(4n-1)}{6}.
\end{aligned}$$

De même, on a $\min(k,j) = \begin{cases} k & \text{si } k \leq j \\ j & \text{si } j \leq k \end{cases}$ d'où

$$\begin{aligned}
\sum_{k=1}^{n}\sum_{j=1}^{n}\min(k,j) &= \sum_{k=1}^{n}\left(\sum_{j=1}^{k-1} j + \sum_{j=k}^{n} k\right) \\
&= \sum_{k=1}^{n}\sum_{j=1}^{k-1} j + \sum_{k=1}^{n}\sum_{j=k}^{n} k \\
&= \sum_{k=1}^{n}\frac{(k-1)k}{2} + \sum_{j=1}^{n}\sum_{k=1}^{j} k \\
&= \frac{1}{2}\sum_{k=1}^{n} k^2 - \frac{1}{2}\sum_{k=1}^{n} k + \sum_{j=1}^{n}\frac{j(j+1)}{2} \\
&= \frac{1}{2}\sum_{k=1}^{n} k^2 - \frac{1}{2}\sum_{k=1}^{n} k + \frac{1}{2}\sum_{j=1}^{n} j^2 + \frac{1}{2}\sum_{j=1}^{n} j \\
&= \sum_{k=1}^{n} k^2 \\
&= \frac{n(n+1)(2n+1)}{6}.
\end{aligned}$$

2. En utilisant plusieurs fois la linéarité de la somme, on a

$$\begin{aligned}
\sum_{k=1}^{n}\sum_{j=1}^{n}|k-j| &= \sum_{k=1}^{n}\left(\sum_{j=1}^{n}\max(k,j)-\min(k,j)\right) \\
&= \sum_{k=1}^{n}\left(\sum_{j=1}^{n}\max(k,j) - \sum_{j=1}^{n}\min(k,j)\right) \\
&= \sum_{k=1}^{n}\sum_{j=1}^{n}\max(k,j) - \sum_{k=1}^{n}\sum_{j=1}^{n}\min(k,j) \\
&= \frac{n(n+1)(4n-1)}{6} - \frac{n(n+1)(2n+1)}{6} \\
&= \frac{n(n+1)(4n-1-(2n+1))}{6} \\
&= \frac{n(n+1)(2n-2)}{6} \\
&= \frac{(n-1)n(n+1)}{3}.
\end{aligned}$$

Exercice 10

Soit $n \in \mathbb{N}^*$. On a

$$\sum_{k=1}^{n} kk! = \sum_{k=1}^{n}(k+1-1)k! = \sum_{k=1}^{n}(k+1)k! - k! = \sum_{k=1}^{n}(k+1)! - k! = (n+1)! - 1.$$

Exercice 11

Soit $n \in \mathbb{N}^*$. On a

$$\prod_{k=1}^{n}(2k-1) = \frac{\prod_{k=1}^{2n} k}{\prod_{k=1}^{n} 2k} = \frac{(2n)!}{2^n \prod_{k=1}^{n} k} = \frac{(2n)!}{2^n n!}.$$

Exercice 12

1. Il s'agit d'une décomposition en éléments simples.
   Supposons qu'il existe trois réels $(a,b,c) \in \mathbb{R}^3$ tels que pour tout $k \in \mathbb{R} \setminus \{0,-1,-2\}$,

   $$\frac{1}{k(k+1)(k+2)} = \frac{a}{k} + \frac{b}{k+1} + \frac{c}{k+2}. \quad (1)$$

   En multipliant (1) par k, on trouve $\frac{1}{(k+1)(k+2)} = a + \frac{bk}{k+1} + \frac{ck}{k+2}$. En évaluant en $k=0$, on trouve $a = \frac{1}{2}$.
   De plus, en faisant tendre k vers $+\infty$, on trouve $a+b+c=0$.
   En multipliant (1) par $k+1$, on trouve $\frac{1}{k(k+2)} = \frac{a(k+1)}{k} + b + \frac{c(k+1)}{k+2}$. En évaluant en $k=-1$, on trouve $b=-1$.
   Enfin, puisque $c = -b-a$, on obtient $c = \frac{1}{2}$.
   Ainsi, pour tout $k \in \mathbb{R} \setminus \{0,-1,-2\}$, $\frac{1}{k(k+1)(k+2)} = \frac{1}{2k} - \frac{1}{k+1} + \frac{1}{2(k+2)}$.
2. Soit $n \in \mathbb{N}^*$.
   On a

   $$u_n = \sum_{k=1}^{n} \frac{1}{k(k+1)(k+2)} = \sum_{k=1}^{n} \frac{1}{2k} - \frac{1}{k+1} + \frac{1}{2(k+2)} = \frac{1}{2}\sum_{k=1}^{n} \frac{1}{k} - \frac{1}{k+1} + \frac{1}{2}\sum_{k=1}^{n} \frac{1}{k+2} - \frac{1}{k+1}.$$

   On reconnaît deux sommes télescopiques et on obtient pour tout $n \in \mathbb{N}^*$,

   $$u_n = \frac{1}{2}\left(1 - \frac{1}{n+1}\right) + \frac{1}{2}\left(\frac{1}{n+2} - \frac{1}{2}\right) = -\frac{1}{2(n+1)(n+2)} + \frac{1}{4}.$$

3. On a $\lim_{n \to +\infty} -\frac{1}{2(n+1)(n+2)} = 0$ donc $\lim_{n \to +\infty} u_n = \frac{1}{4}$.

Exercice 13

Soient $n \in \mathbb{N}^*$ et $q \in \mathbb{R} \setminus \{1\}$.

1. (a) On a

$$\begin{aligned}
T_n &= \sum_{k=1}^{n} kq^{k-1} \\
&= \sum_{k=1}^{n}\sum_{i=1}^{k} q^{k-1} \\
&= \sum_{i=1}^{n}\sum_{k=i}^{n} q^{k-1} \\
&= \sum_{i=1}^{n}\sum_{j=i-1}^{n-1} q^{j} \quad (j = k-1) \\
&= \sum_{i=1}^{n} q^{i-1}\frac{1-q^{n-i+1}}{1-q} \\
&= \frac{1}{1-q}\sum_{i=1}^{n}(q^{i-1} - q^n) \\
&= \frac{1}{1-q}\left(\sum_{i=1}^{n} q^{i-1} - \sum_{i=1}^{n} q^n\right) \\
&= \frac{1}{1-q}\left(\sum_{k=0}^{n-1} q^{k} - nq^n\right) \quad (k = i-1) \\
&= \frac{1}{1-q}\left(\frac{1-q^n}{1-q} - nq^n\right) \\
&= \frac{1-q^n-nq^n(1-q)}{(1-q)^2} \\
&= \frac{1-(n+1)q^n+nq^{n+1}}{(1-q)^2}.
\end{aligned}$$

(b) S_n est une fonction dérivable sur $\mathbb{R}$ et on a pour tout $x \in \mathbb{R}, S'_n(x) = \sum_{k=1}^{n} kx^{k-1}$.

D'autre part, pour tout $x \neq 1$, on a $S_n(x) = \dfrac{1-x^{n+1}}{1-x}$ donc pour tout $x \neq 1$, on a

$$S'_n(x) = \frac{-(n+1)x^n(1-x)+1-x^{n+1}}{(1-x)^2} = \frac{1-(n+1)x^n+nx^{n+1}}{(1-x)^2}.$$

Puisque $T_n = S'_n(q)$, on retrouve bien

$$T_n = \sum_{k=1}^{n} kq^{k-1} = \frac{1-(n+1)q^n+nq^{n+1}}{(1-q)^2}.$$

(c) La fonction S'_n est dérivable sur $\mathbb{R}$ et on a pour tout $x \in \mathbb{R}$ d'une part,

$$S''_n(x) = \sum_{k=2}^{n} k(k-1)x^{k-2}.$$

D'autre part, en dérivant l'autre expression de S_n', on a pour tout $x \neq 1$,

$$\begin{aligned}
S_n''(x) &= \frac{(-n(n+1)x^{n-1} + n(n+1)x^n)(1-x)^2 + 2(1-x)(1-(n+1)x^n + nx^{n+1})}{(1-x)^4} \\
&= \frac{(-n(n+1)x^{n-1} + n(n+1)x^n)(1-x) + 2 - 2(n+1)x^n + 2nx^{n+1}}{(1-x)^3} \\
&= \frac{2 - n(n+1)x^{n-1} + 2(n-1)(n+1)x^n + n(1-n)x^{n+1}}{(1-x)^3}.
\end{aligned}$$

Finalement, on a

$$\begin{aligned}
\sum_{k=0}^{n} k^2 q^k &= q^2 \sum_{k=2}^{n} k(k-1)q^{k-2} + q \sum_{k=1}^{n} kq^{k-1} \\
&= q^2 S_n''(q) + qT_n \\
&= q^2 \frac{2 - n(n+1)q^{n-1} + 2(n-1)(n+1)q^n + n(1-n)q^{n+1}}{(1-q)^3} + q\frac{1-(n+1)q^n + nq^{n+1}}{(1-q)^2}
\end{aligned}$$