(12)

a) $I = \int_1^4 \frac{dx}{\sqrt{x}(1+\sqrt{x})}$

$t = \sqrt{x}$

$dt = \frac{1}{2\sqrt{x}}\,dx$

$I = 2\int_1^4 \frac{1}{2\sqrt{x}(1+\sqrt{x})}\,dx$

$x=1 \to t=1$

$x=4 \to t=2$

$I = 2\int_1^2 \frac{1}{1+t}\,dt = 2\left[\ln(1+t)\right]_{t=1}^{t=2}$

$= 2\ln 3 - 2\ln 2 = \ln\left(\frac{9}{4}\right)$

b) $J = \int_0^{\ln 3} \frac{e^x}{1+e^{2x}}\,dx$

$t = e^x$

$dt = e^x\,dx$

$x=0 \to t=1$

$x=\ln 3 \to t=3$

$e^{2x} = (e^x)^2 = t^2$

$J = \int_1^3 \frac{1}{1+t^2}\,dt$

$J = \text{Arctan}\,3 - \text{Arctan}\,1 = \text{Arctan}\,3 - \frac{\pi}{4}$

c) $K = \int_0^{\pi/2} \sin^3 x\,dx$ avec $t = \cos x$

$dt = -\sin x\,dx$

$K = \int_0^{\pi/2} (-\sin^2 x)(-\sin x)\,dx$

Or $\sin^2 x + \cos^2 x = 1$

$\cos^2 x - 1 = -\sin^2 x$

$K = \int_0^{\pi/2} (\cos^2 x - 1)(-\sin x)\,dx$

$\downarrow dt$

$x=0 \to t = \cos 0 = 1$

$x=\pi/2 \to t = \cos \pi/2 = 0$

$$K = \int_1^0 t^2 - 1 \, dt = \left[\frac{t^3}{3} - t \right]_1^0 = \frac{2}{3}$$

(13) b) $L = \int_1^e \frac{dx}{x(1+\ln x)}$ $\qquad t = \ln x \longrightarrow \begin{cases} x=1 \to t=0 \\ x=e \to t=1 \end{cases}$

$dt = \frac{1}{x} dx$

$$L = \int_0^1 \frac{1}{1+t} dt = \left[\ln(1+t) \right]_{t=0}^{t=1}$$

$$= \ln 2$$

c) $M = \int_0^1 \frac{x}{1+x^4} dx$ $\qquad t = x^4 ? \to dt = 4x^3 dx$

$t = x^2 \to dt = 2x \, dx$

$$M = \frac{1}{2} \int_0^1 \frac{2x}{1+x^4} dx$$

$\begin{cases} x=0 \to t=0 \\ x=1 \to t=1 \end{cases}$

$$M = \frac{1}{2} \int_0^1 \frac{dt}{1+t^2} = \frac{1}{2} (\text{Arctan } 1 - \text{Arctan } 0)$$

$\text{Arctan } 1 \to \frac{\pi}{4}$

$$= \frac{\pi}{8}$$

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(14) $N = \int_0^{\pi} e^{2x} \cos x \, dx$ $\qquad \int u'v = [uv] - \int uv'$

→ Énoncé alternatif : faire 2 I.P.P. pour exprimer N en fonction d'elle-même.

$u'(x) = \cos x$ $\quad v(x) = e^{2x}$

$u(x) = \sin x$ $\quad v'(x) = 2e^{2x}$

$$N = \left[\sin x \, e^{2x} \right]_{x=0}^{x=\pi} - \int_0^{\pi} 2e^{2x} \sin x \, dx$$

$$N = -2 \int_0^{\pi} e^{2x} \sin x \, dx$$

$u'(x) = \sin x$ $\quad v(x) = e^{2x}$

$u(x) = -\cos x$ $\quad v'(x) = 2e^{2x}$

$$N = -2 \left(\left[-\cos x \, e^{2x} \right]_0^{\pi} - \int_0^{\pi} (-\cos x) 2e^{2x} dx \right)$$

$$N = -2 \left(e^{2\pi} + 1 + 2 \int_0^{\pi} \cos x \, e^{2x} dx \right) -$$

$$N = -2e^{2\pi} - 2 - 4N$$

$$5N = -(2+2e^{2\pi})$$

$$N = -\frac{2+2e^{2\pi}}{5}$$

En complexes : $\cos x\, e^{2x} = \mathrm{Re}\left(e^{ix} e^{2x}\right)$

$$= \mathrm{Re}\left(e^{(i+2)x}\right)$$

Primitive de $x \mapsto \cos x\, e^{2x}$:

$$F(x) = \mathrm{Re}\left(\frac{e^{(i+2)x}}{i+2}\right)$$

$$= \mathrm{Re}\left(\frac{-i+2}{1^2+2^2}(\cos x + i\sin x)e^{2x}\right)$$

$$= \frac{e^{2x}}{5}\mathrm{Re}\left((2-i)(\cos x + i\sin x)\right)$$

$$F(x) = \frac{e^{2x}}{5}(2\cos x + \sin x)$$

Ainsi, $\int_0^{\pi} \cos x\, e^{2x}\,dx = \left[\frac{e^{2x}}{5}(2\cos x + \sin x)\right]_0^{\pi} = \frac{2}{5}\left(-e^{2\pi} - 1\right)$

b) $\int_0^x e^{-t}\sin(3t)\,dt$?

$$e^{-x}\sin(3x) = e^{-x}\,\mathrm{Im}\left(e^{i3x}\right)$$

$$= \mathrm{Im}\left(e^{-x} e^{i3x}\right)$$

$$= \mathrm{Im}\left(e^{(-1+3i)x}\right)$$

Primitive : $G(x) = \mathrm{Im}\left(\frac{e^{(-1+3i)x}}{-1+3i}\right)$

$$= \mathrm{Im}\left(\frac{1}{10}(-1-3i)e^{-x} e^{i3x}\right)$$

$$= \mathrm{Im}\left(\frac{e^{-x}}{10} \underbrace{(-1-3i)(\cos 3x + i \sin 3x)}_{e^{i3x}} \right)$$

$$G(x) = \frac{e^{-x}}{10}\left(-\sin(3x) - 3\cos(3x)\right)$$

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a) $\int_0^{\pi/2} \cos^4 x \, dx$

$\cos^4 x = \left(\frac{e^{ix} + e^{-ix}}{2}\right)^4$ $\quad a, b$

$(a+b)^4 \to 1\ 4\ 6\ 4\ 1$ (1 ; 1 1 ; 1 2 1 ; 1 3 3 1)

$a^4 \quad a^3 b \quad a^2 b^2 \quad a b^3 \quad b^4$

$$\cos^4 x = \frac{1}{2^4}\left(e^{i4x} + 4e^{i3x}e^{-ix} + 6e^{i2x}e^{-i2x} + 4e^{ix}e^{-i3x} + e^{-i4x}\right)$$

$$= \frac{1}{2^4}\left(e^{i4x} + 4e^{i2x} + 6 + 4e^{-i2x} + e^{-i4x}\right)$$

$$= \frac{1}{16}\left(\underbrace{e^{i4x} + e^{-i4x}}_{2\cos(4x)} + 4\underbrace{\left(e^{i2x} + e^{-i2x}\right)}_{2\cos(2x)} + 6\right)$$

$$= \frac{1}{8}\cos(4x) + \frac{1}{2}\cos(2x) + \frac{3}{8}$$

Primitive : $F(x) = \frac{1}{8} \times \frac{1}{4}\sin(4x) + \frac{1}{2} \times \frac{1}{2}\sin(2x) + \frac{3}{8}x$

$$F(x) = \frac{1}{32}\sin(4x) + \frac{1}{4}\sin(2x) + \frac{3}{8}x$$

$$\int_0^{\pi/2} \cos^4 x \, dx = \left[F(x)\right]_0^{\pi/2} = \frac{3}{8}\frac{\pi}{2} - 0 = \frac{3\pi}{16}$$

b) $\int_0^{\pi} \sin^2 x \cos^2 x \, dx$?

$\sin(2x) = 2\sin x \cos x$

$\sin x \cos x = \frac{1}{2}\sin(2x)$

$$\sin^2 x \cos^2 x = \frac{1}{4}\sin^2(2x)$$

$\cos(2y) = 1 - 2\sin^2 y$ $\quad$ $y = 2x$:

$\cos(4x) = 1 - 2\sin^2(2x)$

$\times \frac{1}{8}$

$$2\sin^2(2x) = 1 - \cos(4x)$$

$$\underbrace{\frac{1}{4}\sin^2(2x)}_{\sin^2 x \cos^2 x} = \underbrace{\frac{1}{8} - \frac{1}{8}\cos(4x)}_{\text{primitive}}$$

$$\int_0^{\pi} \sin^2 x \cos^2 x \, dx = \left[\frac{1}{8}x - \frac{1}{32}\sin(4x)\right]_0^{\pi}$$

$$= \pi/8$$

$$\sin a \sin b = \frac{1}{2}(\cos(a-b) - \cos(a+b))$$

$$\cos a \cos b = \frac{1}{2}(\cos(a+b) + \cos(a-b))$$

$$\sin a \cos b = \frac{1}{2}(\sin(a+b) + \sin(a-b))$$

$$\cos(a+b) = \cos a \cos b - \sin a \sin b \quad (1)$$

$$\cos(a-b) = \cos a \cos b + \sin a \sin b \quad (2)$$

$$(2) - (1) \rightarrow \cos(a-b) - \cos(a+b) = 2\sin a \sin b$$

$$(1) + (2) \rightarrow \cos(a+b) + \cos(a-b) = 2\cos a \cos b$$

$$(3) \quad \sin(a+b) = \sin a \cos b + \cos a \sin b$$

$$(4) \quad \sin(a-b) = \sin a \cos b - \cos a \sin b$$

$$(3) + (4) : \sin(a+b) + \sin(a-b) = 2\sin a \cos b$$

Primitive de $f(x) = \cos^4(2x)\sin^3(7x)$?

$$\sin a \sin b = \frac{1}{2}(\cos(a-b) - \cos(a+b))$$

$$(2) \quad \cos a \cos b = \frac{1}{2}(\cos(a+b) + \cos(a-b))$$

$$\sin a \cos b = \frac{1}{2}(\sin(a+b) + \sin(a-b))$$

$$(2) \quad \cos(2x) \times \cos(2x) = \frac{1}{2}(\cos(4x) + 1)$$

$$(\cos^2(2x))^2 = \frac{1}{4}\left(\cos^2(4x) + 2\cos(4x) + 1\right)$$

Or $\cos^2(4x) \underset{(2)}{=} \frac{1}{2}(\cos(8x)+1)$

$$\cos^4(2x) = \frac{1}{4}\left(\frac{1}{2}(\cos(8x)+1) + 2\cos(4x) + 1\right)$$
$$= \frac{1}{8}\cos(8x) + \frac{3}{8} + \frac{1}{2}\cos(4x)$$

$\sin(7x) \times \sin(7x) = \ldots$

à finir pour 14h00

Cours IV Equations différentielles linéaires du 1er ordre

1) Equation homogène

I est un intervalle, $a : I \to \mathbb{K}$ est une fonction continue. L'équation :

$$(E_H) \quad y' + a(x)y = 0$$

d'inconnue $y \in \mathcal{C}^1(I, \mathbb{K})$ (i.e. $y : I \to \mathbb{K}$ est dérivable, et y' est continue)

est appelée équation différentielle linéaire homogène du 1er ordre.

Propriété (solutions de cette équation)

Si $A : I \to \mathbb{K}$ est une primitive de a, l'ensemble des solutions de (E_H) est :

$$\left\{ \begin{array}{l} y : I \to \mathbb{K} \\ y : x \mapsto Ce^{-A(x)} \end{array} \;\middle|\; C \in \mathbb{K} \right\}$$

Preuve : Si $y \in \mathcal{C}^1(I, \mathbb{K})$, on note

$$z(x) = e^{A(x)} y(x), \text{ où } z \in \mathcal{C}^1(I, \mathbb{K})$$

et si $x \in I$: $y(x) = z(x) e^{-A(x)}$

d'où : $y'(x) = z'(x) e^{-A(x)} + z(x)\left(-A'(x) e^{-A(x)}\right)$

$$= z'(x) e^{-A(x)} - a(x) z(x) e^{-A(x)}$$

Ainsi, l'équation (E_H) équivaut à :

$$(E_H) \quad y' + a(x) y = 0 \quad \left| \begin{array}{l} z'(x) e^{-A(x)} - a(x) z(x) e^{-A(x)} + a(x) z(x) e^{-A(x)} = 0 \\ \Leftrightarrow \quad z'(x) e^{-A(x)} = 0 \\ \Leftrightarrow \quad z'(x) = 0 \end{array} \right.$$

$$\Leftrightarrow \exists C \in \mathbb{K}, \forall x \in I, z(x) = C$$

$$\Leftrightarrow \exists C \in \mathbb{K}, \forall x \in I, y(x) e^{A(x)} = C$$

$$\Leftrightarrow \exists C \in \mathbb{K}, \forall x \in I, y(x) = C e^{-A(x)}$$

□

Exercice 16 pour 14h00

$$\frac{y'}{y} = -a$$

$$\ln(y) = -A + C^{te}$$

$$y = e^{-A + C^{te}}$$

$$\left(y = e^{C^{te}} e^{-A} \right)$$