exo 8 :

1) Pour $z \in \mathbb{C} \setminus \{4i\}$, on a :

$$\frac{z+2i}{z-4i} \in \mathbb{R} \iff \frac{z+2i}{z-4i} = \overline{\left(\frac{z+2i}{z-4i}\right)} \iff \frac{z+2i}{z-4i} = \frac{\bar{z}-2i}{\bar{z}+4i}$$

$$\iff (z+2i)(\bar{z}+4i) = (\bar{z}-2i)(z-4i)$$

$$\iff z\bar{z} + 2i\bar{z} + 4iz - 8 = z\bar{z} - 2iz - 4i\bar{z} - 8$$

$$\iff 2i(z+\bar{z}) + 4i(z+\bar{z}) = 0$$

$$\iff z+\bar{z} = 0 \iff \underline{z \in i\mathbb{R}}$$

Donc $\mathcal{S}_1 = i\mathbb{R} \setminus \{4i\}$

2) Si $|z| = 1$ alors $|z|^2 = 1$ donc $z\bar{z} = 1$ donc $\bar{z} = \frac{1}{z}$. Dès lors

$$\overline{\left(\frac{z+1}{z-1}\right)} = \frac{\bar{z}+1}{\bar{z}-1} = \frac{\frac{1}{z}+1}{\frac{1}{z}-1} = \frac{1+z}{1-z} = -\frac{z+1}{z-1} \quad \text{donc} \quad \frac{z+1}{z-1} \in i\mathbb{R}$$

3) On a :

$$\frac{z-u\bar{z}}{1-u} \in \mathbb{R} \iff \overline{\left(\frac{z-u\bar{z}}{1-u}\right)} = \frac{z-u\bar{z}}{1-u} \iff \frac{\bar{z}-\bar{u}z}{1-\bar{u}} = \frac{z-u\bar{z}}{1-u}$$

$$\iff (\bar{z}-\bar{u}z)(1-u) = (z-u\bar{z})(1-\bar{u})$$

$$\iff \bar{z} - u\bar{z} - \bar{u}z + u\bar{u}z = z - z\bar{u} - u\bar{z} + u\bar{u}\bar{z}$$

$$\iff z - z + u\bar{u}z - u\bar{u}\bar{z} = 0$$

$$\iff (\bar{z}-z)(1-u\bar{u}) = 0$$

$$\iff z = \bar{z} \text{ ou } u\bar{u} = 1$$

$$\iff z \in \mathbb{R} \text{ ou } |u|^2 = 1$$

$$\iff z \in \mathbb{R} \text{ ou } |u| = 1 \quad (\text{car } |u| \geqslant 0)$$