Oscillateur mécanique

1) $\vec{F} = -k(\ell - \ell_0)\vec{u_z} = -\frac{d}{dz}(E_p)\,\vec{u_z}$, avec $\boxed{E_{p_k} = \frac{1}{2}k(\ell-\ell_0)^2 + \text{cste}}$
(cste nulle car $E_p = 0$ pour $\ell = \ell_0$)

2) $\vec{P} = m\vec{g} \Rightarrow \delta W(\vec{P}) = m\vec{g}\cdot d\vec{\ell} = mg\,d\ell$ vu l'orientation de $\vec{u_z}$
or $\delta W = -dE_p \Rightarrow \boxed{E_{p_p} = -mg\ell + \text{cste}}$ (= nulle car $E_p(0) = 0$)

3) Alors $E_m = \underbrace{\frac{1}{2}m\dot{\ell}^2}_{E_c} \underbrace{- mg\ell + \frac{1}{2}k(\ell-\ell_0)^2}_{E_p}$.

4) Pas de force non conservative (frottements négligés), donc
$E_m = \text{cste}$. $\Rightarrow \frac{dE_m}{dt} = 0 = m\dot{\ell}\ddot{\ell} - mg\dot{\ell} + k\dot{\ell}(\ell - \ell_0) = 0$

$\Leftrightarrow \boxed{\ddot{\ell} + \frac{k}{m}\ell = g + \frac{k}{m}\ell_0}$. (1)

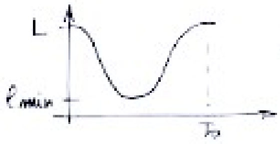
5) On pose $\omega_0 = \sqrt{\frac{k}{m}}$, (1) a pour sol gale $\ell(t) = A\cos\omega_0 t +$
$B\sin\omega_0 t + \ell_0 + \frac{mg}{k}$

A $t = 0$, $\begin{cases} \ell(0) = L \\ \dot{\ell}(0) = 0 \end{cases} \Rightarrow \begin{cases} A + \ell_0 + \frac{mg}{k} = L \\ \omega_0 B = 0 \end{cases} \Rightarrow \begin{cases} A = L - \ell_0 - \frac{mg}{k} \\ B = 0 \end{cases}$

$\Rightarrow \boxed{\ell(t) = \ell_0 + \frac{mg}{k} + \left(L - \ell_0 - \frac{mg}{k}\right)\cos\omega_0 t}$

- on veut $\ell(t) > 0\ \forall t$, or ℓ est min pour $\cos\omega_0 t = -1$
$\Rightarrow \ell_{min} = -L + 2\left(\ell_0 + \frac{mg}{k}\right)$. Il faut donc

$\boxed{2\left(\ell_0 + \frac{mg}{k}\right) > L}$ pour que ℓ ne touche pas 0



avec $T_0 = 2\pi\sqrt{\frac{m}{k}} \overset{AN}{=} 2\pi\sqrt{\frac{0,5}{0,5\,\pi^2}}$
$\Rightarrow \boxed{T_0 = 2\,s}$.

6) $\boxed{E_{pp} = \int_0^\ell -\underbrace{dm}_{}\,g\,z = \int_0^\ell -\frac{m_r}{\ell}dz\,g\,z = -\frac{g\,m_r}{\ell}\frac{\ell^2}{2} = -g\,m_r\frac{\ell}{2}}$
masse d'1 tranche d'épaisseur dz

7) $E_c = \frac{1}{2}\int_0^\ell dm\,v^2$, or $v \overset{\text{énoncé}}{=} Kz$. Comme à
l'extrémité du ressort, $z = \ell$, on a $v(\ell) = K\ell = \dot{\ell}$
$\Rightarrow K = \frac{\dot{\ell}}{\ell} \Rightarrow \boxed{v(z) = \frac{\dot{\ell}}{\ell}z}$.
D'où $E_c = \frac{1}{2}\int_0^\ell \frac{m_r}{\ell}dz\left(\frac{\dot{\ell}}{\ell}z\right)^2 = \frac{1}{2}m_r\frac{\dot{\ell}^2}{\ell^3}\left[\frac{z^3}{3}\right]_0^\ell$

$\Rightarrow \boxed{E_c = \frac{1}{6}m_r\dot{\ell}^2}$

8) On somme les Energies de la masse et du ressort:
$\boxed{E_m = \left(\frac{m}{2} + \frac{m_r}{6}\right)\dot{\ell}^2 + \frac{1}{2}k(\ell-\ell_0)^2 - g\ell\left(m + \frac{m_r}{2}\right)}$

9) {Ressort + masse} subit $\begin{cases} \vec{P} \text{ conservatif} \\ \vec{R} \text{ du plafond de travail nul} \end{cases}$
$\Rightarrow E_m = \text{cste}$
$\Rightarrow \frac{dE_m}{dt} = 0 = \boxed{\left(m + \frac{m_r}{3}\right)\cancel{\dot{\ell}}\ddot{\ell} + k\cancel{\dot{\ell}}(\ell - \ell_0) - g\cancel{\dot{\ell}}\left(m + \frac{m_r}{2}\right)}$ (1)
oscillations à $\omega_1 = \sqrt{\frac{k}{(m + m_r/3)}}$

10) (1) $\Leftrightarrow \ddot{\ell} + \frac{k}{m+m_1/3}\ell = \frac{k\ell_0}{m+m_1/3} + g\left(\frac{m+m_1/2}{m+m_1/3}\right)$

de solution générale

$$\ell = A\cos\omega_1 t + B\sin\omega_1 t + \ell_0 + \frac{g}{k}(m+m_1/2)$$

CI : $\left|\begin{array}{l} \ell(0) = L = A + \ell_0 + g\frac{(m+m_1/2)}{k} \\ \dot{\ell}(0) = 0 = B\omega_1 \end{array}\right.$

Soit $\boxed{\ell(t) = \left[L - \ell_0 - (m+m_1/2)\frac{g}{k}\right]\cos\omega_1 t + \ell_0 + \frac{g}{k}(m+m_1/2)}$

avec $\boxed{\omega_1 = \sqrt{\frac{k}{m+m_1/3}}}$

mouvement harmonique.

On a $T_1 = 2\pi\sqrt{\frac{m+m_1/3}{k}} = 2\pi\sqrt{\frac{m}{k}\left(1+\frac{m_1}{3m}\right)}$

Soit $\boxed{T_1 = T_0\left(1+\frac{m_1}{3m}\right)^{1/2}}$

AN : $T_1 \overset{DL}{\simeq} T_0\left(1+\frac{m_1}{6m}\right) = 2\left(1+\frac{36}{6\times 300}\right) = 2\left(1+\overset{0{,}02}{\frac{6}{300}}\right)$

$\Rightarrow \boxed{T_1 = 2{,}04\,s}$.

11) On veut $\frac{T_1-T_0}{T_0} \le 1\% \Leftrightarrow \frac{m_1}{6m} \le 0{,}01$

$\Leftrightarrow \boxed{m_1 \le 0{,}06m} \overset{AN}{=} \underline{18g}$